Semester 8 Reading Project Report

I. Plasma Instability in Hall Thrusters: Analytical Study

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SEM 8, PHYSICS
UM-DAE-CEBS

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Keywords: Plasma Instabilities ◆ Hall Thrusters ◆ Modified Two Fluid Instability ◆ Ion Acoustic Instability ◆ Electron Cyclotron Drift Instability ◆ Electrostatic Dispersion Relation ◆

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Abstract

Hall Thrusters now plays a prominent role in space propulsion systems owing to its high thrust-to-power ratio, high specific impulse, high efficiency, and simple structure. Partial plasma confinement is fundamental to Hall Thrusters Physics, but that also leads to a variety of instabilities like Modified Two Stream Instability and Electron Cyclotron Drift Instability which takes a toll on efficiency and overall performance of the Thrusters. Although Numerical stimulations have replicated many of these, the cause of these instabilities are still in dark and it is of importance that we have a very solid theoretical understanding of Working Principles. Here, we derive a generic linear dispersion relation for Hall thrusters and attempt to analytically explain these observed instabilities.

Acknowledgement

I give my deepest thanks to my supervisor Dr. Bhooshan Paradkar for introducing me to this beautiful subject and for being such an unlimited source of inspiration. His guidance helped me in all the time of my project and writing of this report.

I am also indebted to my fellow classmates for their careful reading of an early draft of the manuscript and for their many valuable comments and suggestions that have been incorporated in the text to improve the presentation.

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1 Introduction

Partially magnetised plasmas immersed in crossed $E \times B$ fields are employed in electric propulsion systems like Hall thrusters. Such plasmas are prone to a variety of instabilities that impair device operation, particularly instabilities that result in anomalous transport levels that are generally orders of magnitude higher than classical (collisional) transport levels. (Adam, et al., 2008). The nature of anomalous transport (mobility) is yet unknown, however it has been attributed to a number of possible instabilities that may interact to produce the observed levels of anomalous transport.

The presumed structure of this report is to start with a full-fledged derivation of the general Dispersion relation for Hall Thrusters in Chapter 2. Relevant information for the same is either provided within the session of separately discussed in the Appendix. Chapter 3 on Instability Analysis attempts to figure out the dispersion relations by using various Approximations and Asymptotic Analysis. The manuscript concludes with remarks on how to improve on the work.

1.1 Hall Thrusters

Hall thrusters are gridless ion thrusters used in electric space propulsion systems. Xenon ions are extracted from a plasma without grids and accelerated to around 20 km/s in a conventional Hall thruster working in the kW power range (e.g., 300 V, 4 A), and the thrust is on the order of 70 mN. (Boeuf, 2017; Goebel & Katz, 2008). The strong electric field generated in the quasi-neutral plasma of a Hall thruster accelerates ions due to the loss in electron conductivity caused by the presence of a magnetic barrier perpendicular to the passage of electrons from the cathode to the anode.

Permanent magnets provide this external magnetic barrier. The combination of the parallel electric field E and the perpendicular magnetic field B results in a substantial electron drift in the $E \times B$ direction (Hall current). Good confinement of the electrons and an associated drop of electron conductivity can be achieved only if the Hall current does not hit a wall so the $E \times B$ direction must be closed on itself, i.e., must be in the azimuthal direction of a cylindrical configuration, essentially a closed drift device (Zhurin et al., 1999).

In a Hall thruster, the electric field is axial and the magnetic field is radial (see Fig. 1a). Plasma is produced in a channel between two coaxial dielectric cylinders. Electrons are injected through an emissive cathode outside the exhaust plane, with the anode at the channel's end. The magnetic barrier increases the residence time of electrons in the channel, allowing them to ionise the flow of neutral xenon atoms supplied from the anode. Ionization efficiency is very good in Hall thrusters and more than 90% of the gas flow is ionized for applied voltages on the order of 200 V or more.

A major feature of Hall thrusters is that Ionization occurs immediately upstream of the region of high axial electric field, as can be seen in Fig. 1b. Since ions that are essentially unmagnetized, it can easily be removed from the plasma and accelerated by the axial electric field without colliding as the ionisation and acceleration areas are close together and even overlap. The neutral density in the exhaust section of a Hall thruster is very low due to the high ionisation efficiency, therefore electron movement across magnetic field lines cannot be attributed to electron collisions with neutral atoms. (The neutral density is too low by more than a factor of 10 to allow for classical, collisional cross-field transport, see (Boeuf, 2017)).

Despite the fact that Hall thrusters were created over 50 years ago and still used in number of spacecrafts, the electron transport across the magnetic barrier ("anomalous transport) is still a mystery. Anything ranging from Electron collisions with the channel walls and secondary electron emission to instabilities and turbulence could be responsible for this anomalous transport through the magnetic field.

In recent particle simulations of Hall thrusters, the $E \times B$ Electron Drift Instability ($E \times B$ EDI), also known as Electron Cyclotron Drift Instability, has been identified and is a plausible candidate to explain the electron transport across the magnetic field in these devices. The formation of an azimuthal wave and velocity of the order of the ion acoustic velocity, which promotes electron transport across the magnetic field, characterises this instability.

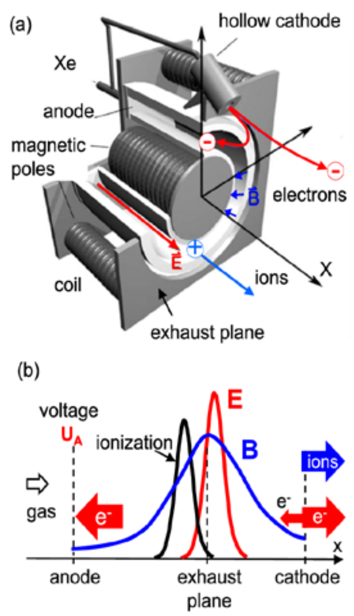

Fig. 1 : a) Schematic of a Hall thruster. b) The curves show, respectively, the axial profiles along the mid channel axis, of the external radial magnetic field, axial electric field, and ionization rate (number of electron-ion pairs generated per unit volume per unit time).

In 2D geometry, a novel form of unstable mode known as the Modified Two-Stream Instability (MTSI) occurs for finite values of the wave number k_z along the magnetic field. The unstable mode resembles the unmagnetized ion sound at larger values of k_z.

1.2 General Approach on Linear Stability Analysis

All plasma phenomena can be explained by combining Maxwell's equations with the Lorentz equation which may be represented by the Vlasov, two-fluid or MHD approximations. **Linearization** or Linear analysis is a field in mathematics of **stability theory** and is a straightforward method applicable to any set of partial differential equations describing a physical system that reveals the physical system's simplest non-trivial, self-consistent dynamical behaviour.

A general procedure for Linear analysis is as follows-

1. By making appropriate physical assumptions, the overall Maxwell-Lorentz equation system is reduced to the simplest set of equations that characterise the phenomena under discussion.
2. We find an equilibrium solution for these set of equations. It is possible that the equilibrium is trivial, with uniform densities, neutral plasma, and zero velocities. However, less trivial equilibria could also be invoked where there are density gradients or flow velocities.

3. Independent variable (usually externally tuneable parameters like electric field) is perturbed and the system of differential equations is solved to get the responses of all the other dependent variables to this prescribed perturbation.
4. Each partial differential equation is re-written with all dependent quantities expanded to first order.

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2 Derivation of the Dispersion Relation

2.1 Lorentz Motion in Uniform E and B

In this session we look at the single particle solution in a uniform E × B field. Motivation for a single particle solution is two-fold (i. Integration over unperturbed trajectories using Method of Characteristics; ii. Constants of Motion for the Distribution Function) and is made more apparent in the following sessions.

We have the Lorentz equation of motion,

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \text{ where } \mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

Solving for $\mathbf{v}$ by making use of linearity of the **source terms**,

i. When $\mathbf{v} \times \mathbf{B} = \mathbf{0}$

$$\dot{\boldsymbol{v}}_{\parallel} = \frac{qE_{\parallel}}{m}$$

ii. When $\mathbf{E} = 0$

$$\mathbf{v}_L = \frac{d\left[\mathbf{r}_L e^{i\Omega(t-t_0)}\right]}{dt}$$

iii. It is quite possible that for some average $\mathbf{v}_d$ the source terms may cancel each other to make the net **zero effect**,

$$\mathbf{E} + \mathbf{v}_d \times \mathbf{B} = 0$$

Taking curl $\boldsymbol{B}$ and rewriting the drift velocity expression,

$$0 = \mathbf{E} \times \mathbf{B} + (\mathbf{v}_d \times \mathbf{B}) \times \mathbf{B} = \mathbf{E} \times \mathbf{B} + (\mathbf{v}_d \cdot \mathbf{B})\mathbf{B} - B^2\mathbf{v}_d$$

$$\mathbf{v}_d = \frac{\mathbf{E} \times \mathbf{B} + (\mathbf{v}_d \cdot \mathbf{B})\mathbf{B}}{B^2} \tag{2.1-1}$$

Now, the component of $\mathbf{v}_d$ parallel to $\boldsymbol{B}$,

$$(\mathbf{v}_d)_{\parallel} = \frac{(\mathbf{E} \times \mathbf{B})_{\parallel} + ((\mathbf{v}_d)_{\parallel} \cdot \mathbf{B})\mathbf{B}}{B^2} = (\mathbf{v}_d)_{\parallel}\hat{\mathbf{B}}$$

which is a trivial solution. While the perpendicular component is,

$$(\mathbf{v}_d)_{\perp} = \frac{(\mathbf{E} \times \mathbf{B})_{\perp} + ((\mathbf{v}_d)_{\perp} \cdot \mathbf{B})\mathbf{B}}{B^2} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

Thus, drift velocity is essentially perpendicular to Magnetic field.

Comments on E × B drift:

1. Independent of the properties of the drifting particle (q, m, v, etc). In particular the drift is in the same direction for both negative charged electrons and positive charged ions.
2. This is essentially a statement that a particle drifts (some average velocity) in such a way to ensure that the electric field seen in its own frame vanishes. This will happen since in the theory of special

relativity the electric field $\boldsymbol{E}'$ observed in a frame moving with velocity $\mathbf{u}$ is $\boldsymbol{E}' = \boldsymbol{E} + \mathbf{u} \times \boldsymbol{B}$[1] and and so $\boldsymbol{E}' = 0$ implies $\mathbf{u}$ is such that $\boldsymbol{E} = -\mathbf{u} \times \boldsymbol{B}$.

3. This does not take into account the relativistic effects and so is valid for $\mathbf{v}_d \ll c$.

Hence, the most general motion of a charged particle in a uniform field would be,

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_d + \mathbf{v}_L = \left(\frac{qE_{\parallel}t}{m} + \mathrm{v}_0\right)\widehat{\mathbf{B}}_{\parallel} + \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{d\left[\mathbf{r}_L e^{i\Omega(t-t_0)}\right]}{dt} \tag{2.1-2}$$

In general, the characteristic scales of each of these components are very different with usually $\mathbf{v}_L$ being considerably fast paced (in terms of length scales gyroradius is much smaller than the wavelength), it is customary to approximate,

$$\mathbf{v} \approx \mathbf{v}_{\parallel} + \mathbf{v}_d$$

which is the motion **about the gyrocenter**.

Quantitatively this approximation fails when,

$$\alpha = \frac{mk^2\phi}{qB^2} \sim 1$$

α is a *non-dimensional measure of the wave amplitude* (McChesney et al., 1987; White et al., 2002).

This is when the wide gap in characteristic scales of slow and fast frequencies vanishes. In such cases, representing the particle's actual location by its gyrocenter is not correct because the particle experiences the wave field at its actual location, not at its gyrocenter.

2.1.1 Single Particle Solutions

The derivation of single particle trajectory is presented in Appendix 4.1 and we summarize the results as here. The velocity is,

$$\mathbf{v}_{\text{unper}}(t') = v_{\parallel}\hat{\mathbf{v}}_{\boldsymbol{z}} + \cos[\omega_{c\sigma}(t'-t)]\mathbf{v}_{\mathbf{x}} + (\sin[\omega_{c\sigma}(t'-t)] - \mathrm{v}_0)\hat{\mathbf{v}}_{\boldsymbol{y}} \tag{2.1-3}$$

- Clearly, $\mathbf{v}_{\parallel}^2$ and $(\mathbf{v}_{\perp}')^2 = (\mathbf{v}_{\perp} - \mathbf{v}_0)^2$ is independent of t and is a Constant of Motion for electrons.

And on integration,

[1] This is the small velocity limit of the Lorentz transformed fields,

$$\begin{aligned}\mathbf{E}' &= \gamma\left(\mathbf{E} + \frac{\mathbf{V}_o}{c} \times \mathbf{B}\right) - \frac{\gamma^2}{\gamma+1}\frac{\mathbf{V}_o}{c}\left(\frac{\mathbf{V}_o}{c} \cdot \mathbf{E}\right) \\ \mathbf{B}' &= \gamma\left(\mathbf{B} - \frac{\mathbf{V}_o}{c} \times \mathbf{E}\right) - \frac{\gamma^2}{\gamma+1}\frac{\mathbf{V}_o}{c}\left(\frac{\mathbf{V}_o}{c} \cdot \mathbf{B}\right)\end{aligned} \xrightarrow{V_0 \ll c} \begin{aligned}\mathbf{E}' &= \mathbf{E} + \frac{\mathbf{V}_o}{c} \times \mathbf{B} \\ \mathbf{B}' &= \mathbf{B}\end{aligned}$$

$$\mathbf{x}(t') = \mathbf{x} + v_{\parallel}(t'-t)\hat{\mathbf{v}}_{\mathbf{z}} + \frac{1}{\omega_{c\sigma}}\{\sin[\omega_{c\sigma}(t'-t)]\mathbf{v}_{\mathbf{x}} + (\cos[\omega_{c\sigma}(t'-t)] - 1)\mathbf{v}_{\mathbf{y}}\} - \mathrm{v}_0(t'-t)\hat{\mathbf{v}}_{\mathbf{y}} \tag{2.1-4}$$

where we have decomposed the velocity along and perpendicular to Magnetic Field ($\hat{z}$)

2.2 Electrostatic ($B_1 = 0$) Instability in Hall Fields

We define some parameters relevant for our problem,

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T_e}{n_e q_e^2}}, \qquad \omega_{p\sigma} = \sqrt{\frac{4\pi n_e \mathrm{q}_\sigma^2}{m^*}}, \qquad r_{L\sigma} = \sqrt{\frac{k_B T_\sigma}{m_\sigma \omega_{c\sigma}^2}}, \qquad \omega_{c\sigma} = \frac{q_\sigma B}{m}, \qquad v_{T\sigma} = \sqrt{\frac{k_B T_\sigma}{m_\sigma}}$$

And the following relation are worth noting,

$$\omega_p \equiv \frac{v_T}{\lambda_D}, \quad \omega_c \equiv \frac{v_T}{r_L}, \qquad r_{L\sigma}^2 \equiv \frac{\omega_{p\sigma}^2 \lambda_{D\sigma}^2}{\omega_{c\sigma}^2}$$

where λ_D is the Debye wavelength, ω_p is the plasma frequency, ω_c the cyclotron frequency, v_T is thermal velocity and r_L is the thermal Lamour radius.

2.2.1 Problem of Interest

The basic scenario is a system as shown in Fig. 2 with a crossed E and B field. The parametric ratios of interest to our problem are:

1. As the typical wavelength is short compared to the thruster radius and because we are interested in Linearized Dispersions, the instability is studied in the local Cartesian frame orthogonal cartesian system with z and y axis characterising the radial and azimuthal angles, respectively. Further the boundary conditions on the global solution would require waves k_y to be periodic. Also, the Plasma is considered to be infinite to avoid any boundary effects.
2. Length scale of B is such that the electrons undergo an $E_0 \times B_0$ drift while the ions being massive are unaffected and remain unmagnetized ($|\omega| \gg \omega_{ci}$). In effect we take $\omega_{ci} \to 0$ or $B \to 0$. This essentially leads to separation of charges along the two perpendicular directions (Fig. 2).

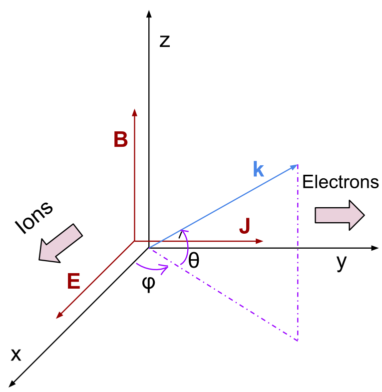

Fig. 2: Schematics of the Hall Thruster Problem

Effect of gradients

The effect of the gradients (density, temperature, and magnetic field) has been studied in the context of collisionless shocks. The density gradient is found to have a negligible effect, while the magnetic field gradient reduces the growth rate by only a few percent. (Priest & Sanderson, 1972). However, the temperature gradient results in an effective increase of the drift velocity v_d, which may be incorporated by a perturbation in the velocity. Nevertheless, here are shall neglect all gradient effects as a first step of simplification.

2.2.2 Distribution function

Following arguments in Sec. 2.2.1, we may construct the distribution function using the Constants of Motions, $\mathbf{v}_\parallel^2$ and $(\mathbf{v}'_\perp)^2 = (\mathbf{v}_\perp - \mathbf{v}_0)^2$ from (2.1-3). Further we assume a gaussian initial distribution.

The normalized distribution function as,

$$f_{\sigma 0}(\mathbf{v}) = \frac{n_{\sigma 0}}{(2\pi v_{T\sigma}^2)^{3/2}} \mathrm{e}^{-\frac{(v_z^2 + v_\perp'^2)}{2\, v_{T\sigma}^2}} \quad \text{where} \quad v_{T\sigma} = \sqrt{k_B T_\sigma / m_\sigma} \tag{2.2-1}$$

2.2.3 Perturbation Equations

Under the perturbations,

$$\vec{B} = \boldsymbol{B} + 0$$
$$\vec{E} = \boldsymbol{E} - \nabla\phi_1$$
$$f_\sigma = f_{\sigma 0} + f_{\sigma 1}$$

where $\boldsymbol{B}$ and $\boldsymbol{E}$ are initial uniform fields, the linearized Vlasov equation takes the form,

$$\frac{\partial f_{\sigma 1}}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v} f_{\sigma 1} + \frac{q_\sigma}{m_\sigma} \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{v} \times \mathbf{B} + \mathbf{E}) f_{\sigma 1} = \frac{q_\sigma}{m_\sigma} \nabla\phi_1 \cdot \frac{\partial f_{\sigma 0}}{\partial \mathbf{v}} \tag{2.2-2}$$

Method of Characteristics

Since,

$$\frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{v} \times \mathbf{B} + \mathbf{E}) = \epsilon_{ijk} \frac{\partial}{\partial \mathrm{v}_j} (\mathbf{v} \times \mathbf{B} + \mathbf{E})_k = \epsilon_{ijk} \frac{\partial}{\partial \mathrm{v}_j} (\epsilon_{klm} \mathrm{v}_l B_m + \mathrm{E}_\mathrm{k}) = \epsilon_{ijk} (\epsilon_{klm} \mathrm{v}_l B_m + \mathrm{E}_\mathrm{k}) \frac{\partial}{\partial \mathrm{v}_j}$$

the LHS of (2.2-2) may be written as,

$$\begin{aligned} \frac{\partial f_{\sigma 1}}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v} f_{\sigma 1} + \frac{q_\sigma}{m_\sigma} \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{v} \times \mathbf{B} + \mathbf{E}) f_{\sigma 1} &= \frac{\partial f_{\sigma 1}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{\sigma 1}}{\partial \mathbf{x}} + \frac{q_\sigma}{m_\sigma} (\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_{\sigma 1}}{\partial \mathbf{v}} + \frac{q_\sigma}{m_\sigma} (\mathbf{E}) \cdot \frac{\partial f_{\sigma 1}}{\partial \mathbf{v}} \\ &\equiv \frac{\mathrm{d}}{\mathrm{d}t} f_{\sigma 1}(\mathbf{x}(t), \mathbf{v}(t), t) \end{aligned}$$

More precisely, $f_{\sigma 1}$ is invariant along the unperturbed single particle trajectory $(\mathbf{x}(t), \mathbf{v}(t))$.

Using this in (2.2-2) and assuming the perturbation quantity is small, we may assume the trajectory $(\mathbf{x}(t), \mathbf{v}(t))$ to be unperturbed,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} f_{\sigma 1}(\mathbf{x}(t), \mathbf{v}(t), t)\right)_{\text{unperturbed trajectory}} = \frac{q_\sigma}{m_\sigma} \nabla\phi_1 \cdot \frac{\partial f_{\sigma 0}}{\partial \mathbf{v}}$$

From Mathematical point of view this is the **Method of Characteristic**[2]. The distribution function as measured along a particle orbit is a constant in time. We can choose to have the distribution function depend on any quantity that is constant along the orbit since it is a constant when measured in the frame following the orbit. (Brueckner & Watson, 1956; Jeans et al., 1915). This provides an effective way for solving the Vlasov problem.

The perturbed distribution function at $\mathbf{x}, \mathbf{v}$ for time t is a result of the sum of all the 'forces' acting over times prior to t calculated along the **unperturbed trajectory of the particle**. The perturbed distribution function is the net outcome of perturbation acting from the very beginning to the present, because that is where the particles at (x, v) were at earlier times and hence where they felt the force. (Landau, 1945)

$$f_{\sigma 1}(\mathbf{x}, \mathbf{v}, t) = \frac{q_\sigma}{m_\sigma} \int_{-\infty}^{t} \mathrm{d}t' \left[\nabla \phi_1 \cdot \frac{\partial f_{\sigma 0}}{\partial \mathbf{v}} \right]_{\mathbf{x}=\mathbf{x}(t'), \mathbf{v}=\mathbf{v}(t')} \tag{2.2-3}$$

where the integration is over the unperturbed trajectory. This is the FORMAL SOLUTION.

Perturbation in Distribution function

Using the perturbation,

$$\phi_1 = \tilde{\phi}_1 \exp \, \mathrm{i}[\mathbf{k} \cdot \mathbf{x}(t) - \omega t]$$

and $f_{\sigma 0}$ as in (2.2-1), we integrate (2.2-5),

$$\begin{aligned} f_{\sigma 1}(\mathbf{x}, \mathbf{v}, t) &= \frac{q_\sigma}{m_\sigma} \int_{-\infty}^{t} \mathrm{d}t' \left[\nabla \phi_1 \cdot \frac{\partial f_{\sigma 0}}{\partial \mathbf{v}} \right]_{\mathbf{x}=\mathbf{x}(t'), \mathbf{v}=\mathbf{v}(t')} \\ &= -\frac{f_{\sigma 0}(\mathbf{v}) q_\sigma \tilde{\phi}_1}{m_\sigma \, v_{T\sigma}^2} \int_{-\infty}^{t} \mathrm{d}t' \, \mathrm{i}\mathbf{k} \exp\left[\mathrm{i}\mathbf{k} \cdot \mathbf{x}(t') - \mathrm{i}\omega t'\right] \cdot (\mathbf{v}(t') - \mathbf{v}_0) \\ &= -\frac{q_\sigma}{m_\sigma \, v_{T\sigma}^2} \tilde{\phi}_1 f_{\sigma 0}(\mathbf{v}) \int_{-\infty}^{t} \mathrm{d}t' \begin{Bmatrix} \exp[-\mathrm{i}\omega t'] \dfrac{\mathrm{d}}{\mathrm{d}t'} \exp[\mathrm{ik} \cdot \mathbf{x}(t')] \\ - \mathrm{i}\mathbf{k} \cdot \mathbf{v}_0 \exp[\mathrm{i}\mathbf{k} \cdot \mathbf{x}(t') - \mathrm{i}\omega t'] \end{Bmatrix} \end{aligned}$$

So that,

$$\begin{aligned} f_{\sigma 1}(\mathbf{x}, \mathbf{v}, t) &= -\frac{q_\sigma}{m_\sigma \, v_{T\sigma}^2} \tilde{\phi}_1 f_{\sigma 0} \begin{Bmatrix} [\exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x}(t') - \mathrm{i}\omega t')]_{-\infty}^{t} + \mathrm{i}\omega I_{phase}(\mathbf{x}, t') \\ -\mathrm{i}\mathbf{k} \cdot \mathbf{v}_0 I_{phase}(\mathbf{x}', t') \end{Bmatrix} \\ &= -\frac{q_\sigma}{m_\sigma \, v_{T\sigma}^2} f_{\sigma 0}(\mathbf{v}) \, \tilde{\phi}_1 \{ \mathrm{e}^{\mathrm{i}\mathbf{k} \cdot \mathbf{x} - \mathrm{i}\omega t} + \mathrm{i}(\omega - \mathbf{k} \cdot \mathbf{v}_0) I_{phase}(\mathbf{x}', t') \} \end{aligned} \tag{2.2-4}$$

In (2.2-4, the lower limit of $t \to -\infty$ relates to the phase in the distant past, and is essentially information about the system's starting state in the distant past. Initial value problems are better described using the Laplace transform method, which is similar to using the Plemelj formula with Fourier

[2] Any hyperbolic partial differential equation can be reduced to a family of ordinary differential equations along which the solution can be integrated from some initial data given on a suitable hypersurface.

transforms (Bellan, 2006). From now on, when it is essential to resolve any singularities in the integrations, Fourier transforms will be utilized in conjunction with the Plemelj formula.

Define $\tau = t' - t$,

$$I_{\text{phase}}(\mathbf{x}, t) = \mathrm{e}^{-\mathrm{i}\omega t}\int_{-\infty}^{t} \mathrm{d}t' \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}(t') - \mathrm{i}\omega(\tau)}$$

$$= \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega t}\int_{-\infty}^{0} \mathrm{d}\tau \exp\left\{\mathrm{i}\left[(k_z v_z - \omega)\tau + \frac{k_\perp v'_\perp}{\omega_{c\sigma}}\{\sin\left[\omega_{c\sigma}\tau + \varphi\right] - \sin\ \varphi\} + k_y v_0 \tau\right]\right.$$

where the Time history of spatial part using (2.1-4),

$$\mathbf{k}\cdot\mathbf{x}(t') = \mathbf{k}\cdot\mathbf{x} + k_z v_z (t'-t) + \frac{k_\perp v'_\perp}{\omega_{c\sigma}}\{\sin[\omega_{c\sigma}(t'-t) + \varphi] - \sin\varphi\} + k_y v_0 (t'-t)$$

with $\mathbf{k}_\perp \cdot \mathbf{v}'_\perp = k_\perp v'_\perp \cos\varphi$, $\mathbf{k}_\perp \cdot \widehat{\mathbf{B}} \times \mathbf{v}'_\perp = k_\perp v'_\perp \sin\varphi$ and $\mathbf{k}_z \cdot \mathbf{v}'_\perp = \mathbf{k}_z \cdot \widehat{\mathbf{B}} \times \mathbf{v}'_\perp = 0$.

Using the Bessel's Identity, (4.5-1),

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}z\sin\theta - \mathrm{i}n\theta}\mathrm{d}\theta, \qquad \mathrm{e}^{\mathrm{i}z\sin\,\theta} = \sum_{n=-\infty}^{\infty} J_n(z)\mathrm{e}^{\mathrm{i}n\theta}$$

we may rewrite the following as,

$$I_{\text{phase}}(\mathbf{x}, t) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega t}\int_{-\infty}^{0} \mathrm{d}\tau \exp\left\{\mathrm{i}\left[(k_z v_z - \omega + k_y v_0)\tau + \frac{k_\perp v'_\perp}{\omega_{c\sigma}}\sin\left[\omega_{c\sigma}\tau + \varphi\right] - \frac{k_\perp v'_\perp}{\omega_{c\sigma}}\sin\varphi\right]\right.$$

$$\Rightarrow\ I_{\text{phase}}(\mathrm{x}, t) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega t}\sum_{n=-\infty}^{\infty} J_n\left(\frac{k_\perp v'_\perp}{\omega_{c\sigma}}\right)\mathrm{e}^{-\frac{\mathrm{i}k_\perp v_\perp \sin\varphi}{\omega_{c\sigma}}}\int_{-\infty}^{0} \mathrm{e}^{\mathrm{i}(k_z v_z - \omega + k_y v_0)\tau + \mathrm{i}n(\omega_{c\sigma}\tau + \varphi)}\mathrm{d}\tau$$

$$= \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathrm{x} - \mathrm{i}\omega t}\mathrm{e}^{-\frac{\mathrm{i}k_\perp v_\perp \sin\varphi}{\omega_{c\sigma}}}\sum_{n=-\infty}^{\infty} J_n\left(\frac{k_\perp v'_\perp}{\omega_{c\sigma}}\right)\frac{\left[\mathrm{e}^{\mathrm{i}(k_z v_z - \omega + k_y v_0)\tau + \mathrm{i}n(\omega_{c\sigma}\tau + \varphi)}\right]_{-\infty}^{0}}{-\mathrm{i}(\omega - k_y v_0 - k_z v_z - n\omega_{c\sigma})}$$

$$I_{\text{phase}}(\mathrm{x}, t) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathrm{x} - \mathrm{i}\omega t}\sum_{m,\ n} J_m\left(\frac{k_\perp v'_\perp}{\omega_{c\sigma}}\right)J_n\left(\frac{k_\perp v'_\perp}{\omega_{c\sigma}}\right)\frac{\mathrm{e}^{-\mathrm{i}m\varphi}\mathrm{e}^{\mathrm{i}n\varphi}}{-\mathrm{i}(\omega - k_y v_0 - k_z v_z - n\omega_{c\sigma})}$$

Using this in (2.2-2),

$$f_{\sigma 1}(\mathbf{x}, \mathbf{v}, t) = -\frac{q_\sigma}{m_\sigma\, v_{T\sigma}^2} f_{\sigma 0}(\mathbf{v})\,\tilde{\phi}_1 \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega t}\left[\begin{array}{c} 1 + (\omega - \mathbf{k}\cdot\mathbf{v}_0)\displaystyle\sum_{m,\ n} J_m\left(\frac{k_\perp v'_\perp}{\omega_{c\sigma}}\right) \\ \times J_n\left(\dfrac{k_\perp v'_\perp}{\omega_{c\sigma}}\right)\dfrac{\mathrm{e}^{\mathrm{i}(n-m)\varphi}}{(k_z v_z + k_y v_0 - \omega + n\omega_{c\sigma})} \end{array}\right] \tag{2.2-5}$$

Perturbation in number density

Since the perturbed number density is,

$$n_{\sigma 1} = \int f_{\sigma_1}\, d\mathbf{v}$$

So that,

$$n_{\sigma 1}=-\frac{q_\sigma}{m_\sigma\, v_{T\sigma}^2}\phi_1(\mathbf{x},t)\,n_{\sigma 0}\left[\begin{array}{c}1+\dfrac{(\omega-\mathbf{k}\cdot\mathbf{v}_0)}{(2\pi)^{3/2}v_{T\sigma}^3}\cdot\displaystyle\sum_{m,n}\frac{1}{\left(k_z v_z+k_y v_0-\omega+n\omega_{c\sigma}\right)}\int_{-\infty}^{\infty}\mathrm{d}v_z\int_0^{2\pi}\mathrm{d}\phi\\ \times\displaystyle\int_0^{\infty}v_\perp'\mathrm{d}v_\perp'\cdot\mathrm{e}^{\mathrm{i}(n-m)\varphi}J_m\left(\frac{k_\perp v_\perp'}{\omega_{c\sigma}}\right)J_n\left(\frac{k_\perp v_\perp'}{\omega_{c\sigma}}\right)\mathrm{e}^{-\frac{(v_z^2+v_\perp'^2)}{2\,v_{T\sigma}^2}}\end{array}\right]$$

$$\Rightarrow\ n_{\sigma 1}=-\frac{q_\sigma}{m_\sigma\, v_{T\sigma}^2}\phi_1(\mathbf{x},t)n_{\sigma 0}\left[\begin{array}{c}1+\dfrac{(\omega-\mathbf{k}\cdot\mathbf{v}_0)}{\sqrt{2\pi}v_{T\sigma}^3}\displaystyle\sum_n\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\frac{v_z^2}{2\,v_{T\sigma}^2}}\mathrm{d}v_z}{\left(k_z v_z+k_y v_0-\omega+n\omega_{c\sigma}\right)}\\ \times\displaystyle\int_0^{\infty}v_\perp'\mathrm{d}v_\perp' J_n^2\left(\frac{k_\perp v_\perp'}{\omega_{c\sigma}}\right)\mathrm{e}^{-\frac{v_\perp'^2}{2\,v_{T\sigma}^2}}\end{array}\right]$$

And so,

$$n_{\sigma 1}=-\frac{q_\sigma}{m_\sigma\, v_{T\sigma}^2}\phi_1(\mathbf{x},t)\,n_{\sigma 0}\left[1+\frac{\omega-k_y v_0}{\sqrt{2}\,k_z v_{T\sigma}}\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}\sum_n I_n(k_\perp^2 r_{L\sigma}^2)\int_{-\infty}^{\infty}\mathrm{d}\xi\frac{\mathrm{e}^{-\xi^2}}{\xi-\alpha_{n\sigma}}\right] \qquad (2.2\text{-}6)$$

where $\alpha_{n\sigma}=\left(\omega-k_y v_{0\sigma}-n\omega_{c\sigma}\right)/\sqrt{2}k_z v_{T\sigma}$ and we have used the Bessel identity ((4.5-4)),

$$\int_0^{\infty}zJ_n^2(\beta z)\mathrm{e}^{-\alpha^2 z^2}\mathrm{d}z=\frac{1}{2\alpha^2}\mathrm{e}^{-\frac{\beta^2}{2\alpha^2}}I_n\left(\frac{\beta^2}{2\alpha^2}\right),$$

And using yet another Bessel's Identity (4.5-2),

$$\sum_{k=-\infty}^{\infty}I_k(x)t^k=\exp\left\{\frac{1}{2}x\left(t+\frac{1}{t}\right)\right\}$$

we have the Ion contribution in the limit $\omega_{ci}\to 0$ so that $\alpha_{n\sigma}\to\alpha_{0\sigma}$,

$$n_{i1}=-\frac{en_0}{m_\sigma v_{T\sigma}^2}\phi_1(\mathbf{x},t)\ \left[1+\frac{\omega}{\sqrt{2}\,k_z v_{T\sigma}}Z\left(\frac{\omega}{\sqrt{2}kv_i}\right)\right] \qquad (2.2\text{-}7)$$

Dispersion relation

Using Eqs. (2.2-6) and (2.2-7) in the Gauss's Law,

$$\nabla^2\phi_1=4\pi(q_i n_i+q_e n_e)$$

We have,

$$\frac{k^2\kappa T_e}{4\pi e^2 n_0}=-\frac{T_e}{T_i}\left[1+\frac{\omega}{\sqrt{2}\,k_z v_{T_i}}Z\left(\frac{\omega}{\sqrt{2}k_z \mathrm{v}_{T_i}}\right)\right]$$
$$-\left[1+\frac{(\omega-k_y v_0)}{\sqrt{2}\,k_z v_{T_e}}\sum_{n=-\infty}^{\infty}\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}\,I_n(k_\perp^2 r_{L\sigma}^2)\,Z\left(\frac{\omega-k_y v_0-n\omega_{ce}}{\sqrt{2}\,k_z v_{T_e}}\right)\right]$$

And, the **warm magnetized plasma** (Doppler shifted electrons) **electrostatic dispersion relation**,

$$T_i\left[1+\frac{k^2 \mathrm{v}_{Te}^2}{\omega_{pe}^2}\right]+T_e\left[1+\frac{\omega}{\sqrt{2}\,k_z v_{T_i}}Z\left(\frac{\omega}{\sqrt{2}k_z \mathrm{v}_{T_i}}\right)\right] + \frac{(\omega-k_y v_0)}{\sqrt{2}\,k_z v_{T_e}}\sum_{n=-\infty}^{\infty} \mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}\, I_n(k_\perp^2 r_{L\sigma}^2)\, Z\left(\frac{\omega-k_y v_0-n\omega_{ce}}{\sqrt{2}\,k_z v_{T_e}}\right)=0 \tag{2.2-8}$$

2.3 Hall Thruster Dispersion Relation:

This may be compactly written as,

$$1+\mathrm{k}^2\lambda_{\mathrm{de}}^2+\frac{\mathrm{T_e}}{\mathrm{T_i}}\big(1+\alpha_{0\mathrm{i}}\mathrm{Z}(\alpha_{0\mathrm{i}})\big)+\alpha_{0\mathrm{e}}\sum_{n=-\infty}^{n=\infty}\beta_{\mathrm{n}}\mathrm{Z}(\alpha_{\mathrm{ne}}) \;=0 \tag{2.3-1}$$

where $\alpha_{0\mathrm{i}} \equiv \omega/\sqrt{2}\mathrm{k}\mathrm{v}_{\mathrm{T_i}}$ and $\beta_{\mathrm{n}} \equiv \mathrm{e}^{-\mathrm{k}_\mathrm{y}^2 r_\mathrm{e}^2}\mathrm{I_n}\big(\mathrm{k}_\mathrm{y}^2 r_\mathrm{e}^2\big)$ with I_{n} being n^{th} order modified Bessel function of the first kind, Z is the Fried - Conte function and $\alpha_{\mathrm{ne}} \equiv \big(\omega-\mathrm{k_y v_0}-\mathrm{n}\omega_{\mathrm{ce}}\big)/\sqrt{2}\mathrm{k}_z v_{\mathrm{T_e}}$ and r_e is electron Lamour Radius.

Many variations of this Dispersion relation have been intensively studied for e.g., in (Ducrocq et al., 2006) who studied the effects of ion plasma with an initial v_x component or in the context of collisionless plasma shocks by (Gary & Sanderson, 1970) who showed that these resonances become less sharp as the angle θ between the magnetic field and the wave vector decreases and that the growth rate remains high in a small solid angle around $\theta = 90^o$ before falling off.

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3 Stability Analysis

3.1 Limiting Cases of Interest

To be draw any insights from relation we look at various limiting cases. A common formalism in the modern literature is to use the Gordeev integral (Gordeev, 1952),

$$g(\Omega, X, Y) = \frac{\Omega}{\sqrt{2Y}} \mathrm{e}^{-X} \sum_{n=-\infty}^{+\infty} \mathrm{Z}\left(\frac{\Omega - n}{\sqrt{2Y}}\right) \mathrm{I}_n(X)$$

As the asymptotic expansion of this has been obtained using different methods, say, the method of steepest descent as in (Paris, 1998) or a method based on the Hadamard expansion of the Gordeev function as in (Schmitt, 1974). We shall proceed with a method similar to (Gary, 1970).

These cases are best expressed when we generalize the dispersion relation in the following manner:

(2.2-6) may be directly substituted in the Gauss's Law to get a more general Dispersion relation,

$$\frac{k^2 \kappa}{4\pi e^2 n_0} + \sum_\sigma \frac{1}{T_\sigma}\left[1 + \alpha_{0\sigma} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}\, I_n(k_\perp^2 r_{L\sigma}^2)\, Z(\alpha_{n\sigma})\right] = 0 \qquad (3.1\text{-}1)$$

This may be rewriting in a different form as follows. Using Bessel's Identity,

$$\mathrm{e}^{-\lambda} \sum_{n=-\infty}^{\infty} I_n(\lambda) = 1$$

we arrive at,

$$1 + \sum_\sigma \frac{1}{k^2 \lambda_{D\sigma}^2}\left[\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2} \sum_{n=-\infty}^{\infty} I_n(k_\perp^2 r_{L\sigma}^2) + \alpha_{0\sigma} \sum_{n=-\infty}^{\infty} \mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}\, I_n(k_\perp^2 r_{L\sigma}^2)\, Z(\alpha_{n\sigma})\right] = 0$$

$$\Rightarrow \quad 1 + \sum_\sigma \frac{\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}}{k^2 \lambda_{D\sigma}^2} \sum_{n=-\infty}^{\infty} I_n(k_\perp^2 r_{L\sigma}^2)\,[1 + \alpha_{0\sigma}\, Z(\alpha_{n\sigma})] = 0$$

And finally using $I_{-n}(z) = -I_n(z)$,

$$1 + \sum_\sigma \frac{\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}}{k^2 \lambda_{D\sigma}^2}\left[I_0(k_\perp^2 r_{L\sigma}^2)\big(1 + \alpha_{0\sigma} Z(\alpha_{0\sigma})\big) + \sum_{n=1}^{\infty} I_n(k_\perp^2 r_{L\sigma}^2)\,[2 + \alpha_{0\sigma}\,\{Z(\alpha_{n\sigma}) + Z(\alpha_{-n\sigma})\}]\right] = 0 \qquad (3.1\text{-}2)$$

- Most general form of **warm magnetized plasma electrostatic dispersion relation** (2.2-8) for Doppler shifted σ species.
- This form is particularly useful for emphasizing the symmetry in n, which are higher order harmonics of cyclotron frequency.
- Note that NO approximations have been made so far. That is to say Eqs. (3.1-1) and (3.1-2) are equivalent in every sense, and either of these may be used for further analysis. A particularly interesting case has been discussed in the Appendix 4.4.

3.1.1 Cold Plasma Limit

We consider the lowest order thermal correction in the cold plasma limit ($|\alpha_0| \equiv |\omega - \mathrm{k_y v_0}|/\mathrm{k}_z \mathrm{v}_T \gg 1$) using the Asymptotic Expansion from Appendix (4.2-3),

$$\begin{aligned} 1+\alpha_0 Z(\alpha_0) = {} & 1+\alpha_0\left\{-\left[\frac{1}{\alpha_0}+\frac{1}{2\alpha_0^3}+\mathcal{O}(\alpha_0^{-5})\right]+\mathrm{i}\sigma\sqrt{\pi}\mathrm{e}^{-\alpha_0^2}\right\} \\ & \approx -\frac{1}{2\alpha_0^2}+\mathcal{O}(\alpha_0^{-4})+\mathrm{i}\sigma\alpha_0\sqrt{\pi}\,\mathrm{e}^{-\alpha_0^2} \sim -\frac{1}{2\alpha_0^2} \end{aligned} \tag{3.1-3}$$

Additionally, $|\alpha_n| = |(\omega - k_y v_0 - n\omega_{c\sigma})/k_\parallel v_T| \gg 1$, for $n \neq 0$; this corresponds to assuming that the wave frequency is not too close to a cyclotron resonance, so that,

$$\begin{aligned} 2+\alpha_0[Z(\alpha_n)+Z(\alpha_{-n})] & \approx 2+\alpha_0\left[-\frac{1}{\alpha_n}-\frac{1}{\alpha_{-n}}+\mathcal{O}(\alpha_n^{-3})+\mathrm{i}\sigma\sqrt{\pi}\left(\mathrm{e}^{-\alpha_n^2}+\mathrm{e}^{-\alpha^2_{-n}}\right)\right] \\ & \approx 2-\frac{\omega-k_y v_0}{\omega-k_y v_0-n\omega_{c\sigma}}-\frac{\omega-k_y v_0}{\omega-k_y v_0+n\omega_{c\sigma}}+\mathcal{O}(\alpha_n^{-2}) \\ & \approx -\frac{2n^2\omega_{c\sigma}^2}{(\omega-k_y v_0)^2-n^2\omega_{c\sigma}^2} \end{aligned} \tag{3.1-4}$$

- Readers might have noticed the leading order error in expansion is of $\mathcal{O}(\alpha_0^{-4})$ in (3.1-3) while it is of $\mathcal{O}(\alpha_n^{-2})$ in (3.1-4). This is done for mathematical consistency as discussed in Appendix 4.4.

  And so, from (3.1-2) we have,

$$1-\sum_\sigma \frac{\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}}{k^2\lambda_{D\sigma}^2}\left[I_0(k_\perp^2 r_{L\sigma}^2)\frac{1}{2}\left(\frac{k_\parallel v_{T\sigma}}{\omega-k_y v_{\sigma 0}}\right)^2+\sum_{n=1}^{\infty} I_n(k_\perp^2 r_{L\sigma}^2)\left(\frac{2n_\sigma^2\omega_{c\sigma}^2}{(\omega-k_y v_{\sigma 0})^2-n_\sigma^2\omega_{c\sigma}^2}\right)\right]=0$$

Define the doppler shifted wave frequency as $\omega_\sigma = \omega - k_y v_{\sigma 0}$ to get,

$$1-\sum_\sigma \frac{\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}}{k^2\lambda_{D\sigma}^2}\left[\frac{(k_\parallel v_{T\sigma})^2}{2\omega_\sigma{}^2} I_0(k_\perp^2 r_{L\sigma}^2)-\sum_{n=1}^{\infty} I_n(k_\perp^2 r_{L\sigma}^2)\left(\frac{2n_\sigma^2\omega_{c\sigma}^2}{\omega_\sigma{}^2-n_\sigma^2\omega_{c\sigma}^2}\right)\right]=0 \tag{3.1-5}$$

Thus, the cold plasma dispersion relation in our scenario is,

$$1-\frac{k_z^2}{k^2}\frac{\omega_{pi}^2}{\omega^2}-\frac{\mathrm{e}^{-k_\perp^2 r_{Le}^2}}{k^2\lambda_{De}^2}\left[\frac{k_z^2 v_{T_e}^2\, I_0(k_\perp^2 r_{Le}^2)}{(\omega-k_y v_0)^2}-\sum_{n=1}^{\infty}\frac{2n_e^2\omega_{ce}^2}{(\omega-k_y v_0)^2-n_e^2\omega_{ce}^2} I_n(k_\perp^2 r_{Le}^2)\right]=0 \tag{3.1-6}$$

- Notice that the Landau damping appears both at the wave frequency ω and also at cyclotron harmonics, i.e., in the vicinity of $n\omega_{c\sigma}$. We shall attempt an analysis of this limit in the following chapter.

3.1.2 High T Plasma Limit

We consider the lowest order thermal correction in the hot plasma limit ($|\alpha_0| \ll 1$) using the Asymptotic Expansion from Appendix (4.2-3),

$$1 + \alpha_0 Z(\alpha_0) = 1 + \alpha_0 \left\{ -\mathrm{i}\sqrt{\pi}\, e^{-\alpha_0{}^2} - 2\alpha_0 \left[1 - \frac{2\alpha_0{}^2}{3} + \cdots \right] \right\} \approx 1 - \mathrm{i}\alpha_0 \sqrt{\pi}\, e^{-\alpha_0{}^2} - 2\alpha_0^2$$
$$\approx 1 - 2\alpha_0^2 - \mathrm{i}\alpha_0\sqrt{\pi}$$

Additionally, $|\alpha_n| = |(\omega - k_y v_0 - n\omega_{c\sigma})/k_\parallel v_{T\sigma}| \ll 1$, for $n \neq 0$; this corresponds to the case when $v_{T\sigma} \gg 1$, so that,

$$2 + \alpha_0[Z(\alpha_n) + Z(\alpha_{-n})] \approx 2 + \alpha_0 \left\{ \mathrm{i}\sqrt{\pi}\, e^{-\alpha_n{}^2} - 2\alpha_0 \left[1 - \frac{2\alpha_n{}^2}{3} \right] + \mathrm{i}\sqrt{\pi}\, e^{-\alpha_{-n}{}^2} - 2\alpha_0 \left[1 - \frac{2\alpha_{-n}{}^2}{3} \right] \right\}$$
$$\approx 2 + 2\alpha_0 \left[\mathrm{i}\sqrt{\pi} - 2\alpha_0 \left(1 - \frac{(\alpha_n{}^2 + \alpha_{-n}{}^2)}{3} \right) \right]$$

And so, using (3.1-2) we have the Hot Plasma limit for the dispersion relation,

$$1 + \sum_\sigma \frac{\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}}{k^2 \lambda_{D\sigma}^2} \left[I_0(k_\perp^2 r_{L\sigma}^2)(1 - 2\alpha_{0\sigma}^2 - \mathrm{i}\alpha_{0\sigma}\sqrt{\pi}) + 2\sum_{n=1}^{\infty} I_n(k_\perp^2 r_{L\sigma}^2) \left\{ 1 - 2\alpha_{0\sigma}^2 \left(1 - \frac{(\alpha_{n\sigma}{}^2 + \alpha_{-n\sigma}{}^2)}{3} \right) + \mathrm{i}\alpha_{0\sigma}\sqrt{\pi} \right\} \right] = 0$$

Analyzing this limit is very tricky not only because of the complex dispersion relation but mainly because of the fact that $|\alpha_n| = |(\omega - k_y v_0 - n\omega_{c\sigma})/k_\parallel v_{T\sigma}| \ll 1$ happens due to not just $v_{T\sigma} \to \infty$, but also when $|\omega - k_y v_0| \sim |n\omega_{c\sigma}|$ which leads to strong coupling of electron and ion modes.

Also, in this limit since ions are highly energetic, they can no longer be treated as unmagnetized $(\omega \ll \omega_{ci})$ which is a necessary requirement in Hall Thrusters. This leads us to the next limiting case.

3.1.3 Semi Cold Plasma Limit

A limit that may be of particular importance to Hall Thruster Instabilities is the case when $T_e \gg T_i$ leading to $\alpha_{ne} \ll 1$ and $\alpha_{ni} \gg 1$ expansions in the Dispersion relation.

The last two limits have been mentioned for completeness and an in-depth analysis is beyond the scope of this report.

3.2 Kinds of Instability

We focus our attention on two basic modes of Instability in the regime of $k_y r_{Le} \ll 1$ or small perturbation wavevector.

3.2.1 Modified Two Stream Instability –

Two Stream Instabilities arise due to interaction among different species that are separated by drift direction and velocity. For the case where $\omega - k_y v_0 \ll \omega_{ce}$ (ions are unmagnetized), $k_y r_{Le} \ll 1$ (so that only the lowest-order finite Larmor radius terms are retained), For $k_y r_{Le} \ll 1$, we have the following asymptotic expansion (4.5-3),

$$\operatorname*{Lim}_{\lambda \ll 1} I_n(\lambda) = \frac{1}{n!}\left(\frac{\lambda}{2}\right)^n \text{ where } \lambda \equiv k_\perp^2 r_{L\sigma}^2$$

Using (3.1-5),

$$k^2 - \sum_\sigma \frac{(1 - k_\perp^2 r_{L\sigma}^2)}{\lambda_{D\sigma}^2}\left[\left(\frac{k_z^2 v_{T_\sigma}^2}{{\omega_\sigma}^2}\right) + I_1(k_\perp^2 r_{L\sigma}^2)\left(\frac{2\omega_{c\sigma}^2}{{\omega_\sigma}^2 - \omega_{c\sigma}^2}\right) + I_2(k_\perp^2 r_{L\sigma}^2)\left(\frac{8\omega_{c\sigma}^2}{{\omega_\sigma}^2 - 4\omega_{c\sigma}^2}\right)\right] = 0$$

We have,

$$k^2 - \sum_\sigma (1 - k_y^2 r_{L\sigma}^2)\left[\frac{1}{\lambda_{D\sigma}^2}\left(\frac{k_z^2 v_{T_\sigma}^2}{{\omega_\sigma}^2}\right) + \frac{k_y^2 r_{L\sigma}^2}{2\,\lambda_{D\sigma}^2}\left(\frac{2\omega_{c\sigma}^2}{{\omega_\sigma}^2 - \omega_{c\sigma}^2}\right) + \frac{1}{2!}\frac{k_y^4 r_{L\sigma}^4}{4\,\lambda_{D\sigma}^2}\left(\frac{8\omega_{c\sigma}^2}{{\omega_\sigma}^2 - 4\omega_{c\sigma}^2}\right)\right] = 0$$

$$\Rightarrow\ k_y^2 + k_z^2 - \sum_\sigma \left\{\left[\frac{1}{\lambda_{D\sigma}^2}\left(\frac{k_z^2 v_{T_\sigma}^2}{{\omega_\sigma}^2}\right) + \frac{k_y^2 r_{L\sigma}^2}{\lambda_{D\sigma}^2}\left(\frac{\omega_{c\sigma}^2}{{\omega_\sigma}^2 - \omega_{c\sigma}^2}\right) + \frac{k_y^4 r_{L\sigma}^4}{\lambda_{D\sigma}^2}\left(\frac{\omega_{c\sigma}^2}{{\omega_\sigma}^2 - 4\omega_{c\sigma}^2}\right)\right]\right.$$
$$\left. + \left[\frac{1}{\lambda_{D\sigma}^2}\left(\frac{k_z^2 v_{T_\sigma}^2 k_y^2 r_{L\sigma}^2}{{\omega_\sigma}^2}\right) + \frac{k_y^4 r_{L\sigma}^4}{\lambda_{D\sigma}^2}\left(\frac{\omega_{c\sigma}^2}{{\omega_\sigma}^2 - \omega_{c\sigma}^2}\right) + \frac{k_y^6 r_{L\sigma}^6}{\lambda_{D\sigma}^2}\left(\frac{\omega_{c\sigma}^2}{{\omega_\sigma}^2 - 4\omega_{c\sigma}^2}\right)\right]\right\} = 0$$

Ignoring terms of $\mathcal{O}(k_y^6 r_{L\sigma}^6)$ since order above $k_y^4 r_{L\sigma}^4$ is not solved consistently,

$$k_y^2\left(1 - \sum_\sigma \frac{r_{L\sigma}^2}{\lambda_{D\sigma}^2}\left(\frac{\omega_{c\sigma}^2}{{\omega_\sigma}^2 - \omega_{c\sigma}^2}\right)\right) + k_z^2\left(1 - \sum_\sigma \frac{1}{\lambda_{D\sigma}^2}\left(\frac{v_{T_\sigma}^2}{{\omega_\sigma}^2}\right)\right) - k_y^4 \sum_\sigma \left[\frac{r_{L\sigma}^4}{\lambda_{D\sigma}^2}\left(\frac{3\omega_{c\sigma}^2}{({\omega_\sigma}^2 - 4\omega_{c\sigma}^2)({\omega_\sigma}^2 - \omega_{c\sigma}^2)}\right)\right]$$
$$+ \sum_\sigma \left[\frac{k_y^2 k_z^2}{\lambda_{D\sigma}^2}\left(\frac{v_{T_\sigma}^2 r_{L\sigma}^2}{{\omega_\sigma}^2}\right)\right] = 0$$

Thus,

$$k_y^2\left(1 - \sum_\sigma \frac{r_{L\sigma}^2}{\lambda_{D\sigma}^2}\left(\frac{\omega_{c\sigma}^2}{{\omega_\sigma}^2 - \omega_{c\sigma}^2}\right)\right) + k_z^2\left(1 - \sum_\sigma \frac{1}{\lambda_{D\sigma}^2}\left(\frac{v_{T_\sigma}^2}{{\omega_\sigma}^2}\right)\right) - k_y^4 \sum_\sigma \left[\frac{r_{L\sigma}^4}{\lambda_{D\sigma}^2}\left(\frac{3\omega_{c\sigma}^2}{({\omega_\sigma}^2 - 4\omega_{c\sigma}^2)({\omega_\sigma}^2 - \omega_{c\sigma}^2)}\right)\right]$$
$$+ \sum_\sigma \left[\frac{k_y^2 k_z^2}{\lambda_{D\sigma}^2}\left(\frac{v_{T_\sigma}^2 r_{L\sigma}^2}{{\omega_\sigma}^2}\right)\right] = 0$$

which may be rewritten as,

$$k_y^2\begin{pmatrix} 1 - \dfrac{r_{Le}^2}{\lambda_{De}^2}\left(\dfrac{\omega_{ce}^2}{{\omega_e}^2 - \omega_{ce}^2}\right) \\ -\dfrac{r_{Li}^2}{\lambda_{Di}^2}\left(\dfrac{\omega_{ci}^2}{{\omega_i}^2 - \omega_{ci}^2}\right) \end{pmatrix} + k_z^2\begin{pmatrix} 1 - \dfrac{1}{\lambda_{De}^2}\left(\dfrac{v_{T_e}^2}{{\omega_e}^2}\right) \\ -\dfrac{1}{\lambda_{Di}^2}\dfrac{v_{T_i}^2}{\omega^2} \end{pmatrix}$$
$$- k_y^4\begin{bmatrix} \dfrac{r_{Le}^4}{\lambda_{De}^2}\left(\dfrac{3\omega_{ce}^2}{({\omega_e}^2 - 4\omega_{ce}^2)({\omega_\sigma}^2 - \omega_{ce}^2)}\right) \\ + \dfrac{r_{Li}^4}{\lambda_{Di}^2}\left(\dfrac{3\omega_{ci}^2}{({\omega_i}^2 - 4\omega_{ci}^2)({\omega_i}^2 - \omega_{ci}^2)}\right) \end{bmatrix} + k_y^2 k_z^2\begin{bmatrix} \dfrac{1}{\lambda_{De}^2}\left(\dfrac{v_{T_e}^2 r_{Le}^2}{{\omega_e}^2}\right) \\ + \dfrac{1}{\lambda_{Di}^2}\dfrac{v_{T_i}^2 r_{Li}^2}{{\omega_i}^2} \end{bmatrix} = 0 \quad (3.2\text{-}1)$$

For the current scenario ($v_{i0} = 0,\ \omega_{ci}^2 \ll \omega^2 \ll \omega_{ce}^2$),

$$k_y^2\left(1+\frac{r_{Le}^2}{\lambda_{De}^2}-\frac{r_{Li}^2}{\lambda_{Di}^2}\left(\frac{\omega_{ci}^2}{\omega^2}\right)\right)+k_z^2\left(1-\frac{1}{\lambda_{De}^2}\left(\frac{v_{T_e}^2}{\left(\omega-k_y v_{e0}\right)^2}\right)-\frac{1}{\lambda_{Di}^2}\frac{v_{T_i}^2}{\omega^2}\right)$$
$$-k_y^4\left[\frac{r_{Le}^4}{\lambda_{De}^2}\left(\frac{3}{(4\omega_{ce}^2)}\right)+\frac{r_{Li}^4}{\lambda_{Di}^2}\left(\frac{3\omega_{ci}^2}{(\omega^4)}\right)\right]+k_y^2k_z^2\left[\frac{1}{\lambda_{De}^2}\left(\frac{v_{T_e}^2 r_{Le}^2}{\left(\omega-k_y v_{e0}\right)^2}\right)+\frac{1}{\lambda_{Di}^2}\frac{v_{T_i}^2 r_{Li}^2}{\omega^2}\right]=0$$

To fourth order,

$$k_y^2\left(1+\frac{\omega_{pe}^2}{\omega_{ce}^2}-\frac{\omega_{pi}^2}{\omega^2}\right)+k_z^2\left(1-\frac{\omega_{pe}^2}{\left(\omega-k_y v_{e0}\right)^2}-\frac{\omega_{pi}^2}{\omega^2}\right)-k_y^4\left[\frac{3}{4\omega_{ce}^2}\frac{r_{Le}^4}{\lambda_{De}^2}+\frac{3\omega_{ci}^2}{\omega^4}\frac{r_{Li}^4}{\lambda_{Di}^2}\right]$$
$$+k_y^2k_z^2\left[\frac{\omega_{pe}^2 r_{Le}^2}{\left(\omega-k_y v_{e0}\right)^2}+\frac{\omega_{pi}^2 r_{Li}^2}{\omega^2}\right]=0 \quad (3.2\text{-}2)$$

To second order,

Since $\omega_p \equiv v_T/\lambda_D$ and $\omega_c \equiv v_T/r_L$,

$$k_y^2\left(1+\frac{\omega_{pe}^2}{\omega_{ce}^2}-\frac{\omega_{pi}^2}{\omega^2}\right)+k_z^2\left(1-\frac{\omega_{pe}^2}{\left(\omega-k_y v_{e0}\right)^2}-\frac{\omega_{pi}^2}{\omega^2}\right)=0 \quad (3.2\text{-}3)$$

Rewriting this as,

$$1-\frac{\omega_{pi}^2}{\omega^2}-\frac{\omega_{pe}^2 k_z^2}{\left(\omega-k_y v_0\right)^2 k^2}+\frac{\omega_{pe}^2 k_y^2}{\omega_{ce}^2 k^2}=0$$

This may be thought of as coupling between $\omega^2 \sim \omega_{pi}^2$, and the doppler shifted coupling of electrons,

$$\left(\omega-k_y v_0\right)^2 \sim \frac{\omega_{pe}^2 k_z^2}{k_y^2\left(\omega_{pe}^2/\omega_{ce}^2\right)+k^2}$$

which reduces to the lower electron hybrid mode for $k^2 \sim k_y^2$,

$$\left(\omega-k_y v_0\right)^2 \sim \frac{k_z^2}{k_y^2}\frac{\omega_{pe}^2}{\left(\omega_{pe}^2/\omega_{ce}^2\right)+1} \equiv \frac{k_z^2}{k_y^2}\,\omega_{LH}^2$$

The reactive modified two-stream instability changes into the dissipative ion-acoustic instability as the ratio $k_z\,/\,k_y$ is increased further. Although this condition intermediate condition is very hard to be realized analytically, this regime has been looked into by numerical means. It has also been shown that for a large enough $v_{\rm th}k_z\,/\,\omega_{ce}$ ratio the dispersion relation approaches that of an ion acoustic wave[3] despite the fact that the wave vector is almost orthogonal to the magnetic field (Gary & Sanderson, 1970).

[3] This observation is of particular interest because ion acoustic modes are essentially sound/ longitudinal wave perturbations and their existence despite the fact that the wavevector is almost perpendicular to perturbation ($\nabla\phi$) requires special attention.

3.2.2 Electron Cyclotron Drift Instability –

Due to resonances of the ion-acoustic mode and electron cyclotron harmonics, the electron $E \times B$ flow relative to unmagnetized ions leads to the Electron Cyclotron Drift Instability (ECDI). ECDI is expected to be a key element in creating an electron flow parallel to the background E field at a rate that far exceeds what classical collision theory predicts. Undesirable plasma fluxes towards the walls of $E \times B$ devices may also be a result from such anomalous transport (Wang et al., 2021).

The defining condition for this instability is when $k_{\parallel}/k_{\perp} \ll 1$. Now this limit also leads to the cold plasma limit and so from (3.1-6),

$$1 - \frac{k_z^2}{k^2}\frac{\omega_{pi}^2}{\omega^2} - \frac{e^{-k_\perp^2 r_{Le}^2}}{k^2\lambda_{De}^2}\left[\frac{k_z^2 v_{T_e}^2\, I_0(k_\perp^2 r_{Le}^2)}{(\omega - k_y v_0)^2} - \sum_{n=1}^{\infty}\frac{2n_e^2\omega_{ce}^2}{(\omega - k_y v_0)^2 - n_e^2\omega_{ce}^2} I_n(k_\perp^2 r_{Le}^2)\right] = 0$$

We take $k_z \sim 0$,

$$k_y^2\left(1 - \left(\frac{\omega_{pe}^2}{{\omega_e}^2 - \omega_{ce}^2}\right) - \left(\frac{\omega_{pi}^2}{{\omega_i}^2 - \omega_{ci}^2}\right)\right) - k_y^4\left[r_{Le}^2\left(\frac{3\omega_{pe}^2}{({\omega_e}^2 - 4\omega_{ce}^2)({\omega_e}^2 - \omega_{ce}^2)}\right) + r_{Li}^2\left(\frac{3\omega_{pi}^2}{({\omega_i}^2 - 4\omega_{ci}^2)({\omega_i}^2 - \omega_{ci}^2)}\right)\right] = 0$$

To fourth order,

In the strict limit of $\omega_{ci}^2 \ll \omega^2$,

$$k_y^2\left(1 - \left(\frac{\omega_{pe}^2}{{\omega_e}^2 - \omega_{ce}^2}\right) - \left(\frac{\omega_{pi}^2}{{\omega_i}^2}\right)\right) - k_y^4\left[r_{Le}^2\left(\frac{3\omega_{pe}^2}{({\omega_e}^2 - 4\omega_{ce}^2)({\omega_e}^2 - \omega_{ce}^2)}\right) + 3r_{Li}^2\frac{\omega_{pi}^2}{{\omega_i}^4}\right] = 0$$

To second order,

$$k_y^2\left(1 - \frac{\omega_{pe}^2}{(\omega - k_y v_{e0})^2 - \omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2}\right) = 0 \tag{3.2-4}$$

The instability is a result of reactive coupling of the electron (Doppler shifted) upper hybrid mode $(\omega - kv_0)^2 = \omega_{pe}^2 + \omega_{ce}^2$ with the short wavelength ion oscillations $\omega^2 = \omega_{pi}^2$. The contribution of higher $m > 1$ harmonics in (3.1-2) (which are absent for $T_e = 0$) grows with electron temperature and has the maximum at shorter wavelengths $k_\perp^2 r_{Le}^2 \approx 1$ due to the $e^{-k_\perp^2 r_{Le}^2}\, I_n(k_\perp^2 r_{Le}^2)$ factors. (Lashmore, D. et al., 1973)

3.3 Further prospects

- Numerical Stimulation and matching of the above analytical insights.

- 4^{th} and higher order expansions are not consistently solved, since we started with a linearized Vlasov Perturbation. Moving forward an important task would be to analyse the second order dispersion relation and understanding how these and new modes intertwine in the complex situation.
- The dispersion equation for this problem (2.2-8) was derived assuming that the particles had Maxwellian or shifted Maxwellian velocity distributions. The dissipative instabilities described in this manuscript are all very sensitive to the form of the velocity distribution and may be have very different consequences depending on the distribution profile.
- We have neglected the gradient effects for low-β plasma (ratio of electron pressure to magnetic pressure) case where the $E_0 \times B_0$ drift is found to dominant. If the gradient effects are put in, then the electromagnetic terms should also be included.

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4 Appendix

4.1 First Principal derivation of Velocity –

We start with the Lorentz equation of Motion for a charged particle,

$$\dot{\mathbf{v}} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \frac{q}{m}\left(E\hat{x} + v_y B\hat{x} - v_x B\hat{y}\right)$$

$$\dot{v}_x = \frac{qB}{m}\left(\frac{E}{B} + v_y\right), \qquad \dot{v}_y = -\frac{qB}{m} v_x, \qquad \dot{v}_z = 0 \tag{4.1-1}$$

And,

$$\ddot{v}_x = -\left(\frac{qB}{m}\right)^2 v_x, \qquad \ddot{v}_y = -\left(\frac{qB}{m}\right)^2 \left(\frac{E}{B} + v_y\right), \qquad v_z = v_{\parallel} \tag{4.1-2}$$

$$\Rightarrow \qquad v_x(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t} + A_3, \qquad v_y(t) = B_1 e^{i\omega t} + B_2 e^{-i\omega t} + B_3$$

From (4.1-2),

$$v_x(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t} + A_3 = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$$
$$v_y(t) = B_1 e^{i\omega t} + B_2 e^{-i\omega t} + B_3 = B_1 e^{i\omega t} + B_2 e^{-i\omega t} - E/B$$

From (4.1-1),

$$B_1 = -A_2, \qquad B_2 = A_1$$
$$v_x(t') = A_1 e^{i\omega t'} + A_2 e^{-i\omega t'}, \qquad v_y(t') = -A_2 e^{i\omega t'} + A_1 e^{-i\omega t'} - E/B$$

Boundary conditions

Take $\mathbf{v}(0) = \mathbf{v_x} + v_0 \hat{\mathbf{v}}_{\mathbf{y}} + v_{\parallel} \hat{\mathbf{v}}_{\mathbf{z}}$,

$$A_1 = A_2 \quad \Rightarrow \quad \mathbf{v}_{\text{unper}}(t') = v_{\parallel} \hat{\mathbf{v}}_{\mathbf{z}} + \cos[\omega_{c\sigma}(t')]\mathbf{v_x} - (\sin[\omega_{c\sigma}(t')] + \mathrm{v}_0)\mathbf{v_y}$$

where $\mathrm{v}_0 = -E/B$.

Notice that $\mathbf{v}_{\text{unper}}(t' - t)$ at $t' = t$ has the same boundary conditions as above,

$$\mathbf{v}_{\text{unper}}(t' - t) \equiv \mathbf{v}_{\text{unper}}(t') = v_{\parallel} \hat{\mathbf{v}}_{\mathbf{z}} + \cos[\omega_{c\sigma}(t' - t)]\mathbf{v_x} - (\sin[\omega_{c\sigma}(t' - t)] + \mathrm{v}_0)\mathbf{v_y}$$

Velocity is,

$$\mathbf{v}_{\text{unper}}(t') = v_{\parallel} \hat{\mathbf{v}}_{\mathbf{z}} + \cos[\omega_{c\sigma}(t' - t)]\mathbf{v_x} + (\sin[\omega_{c\sigma}(t' - t)] - \mathrm{v}_0)\hat{\mathbf{v}}_{\mathbf{y}} \tag{4.1-3}$$

And on integration,

$$\mathbf{x}(t') = \mathbf{x} + v_{\parallel}(t' - t)\hat{\mathbf{v}}_{\mathbf{z}} + \frac{1}{\omega_{c\sigma}}\left\{\sin\left[\omega_{c\sigma}(t' - t)\right]\mathbf{v_x} + \left(\cos\left[\omega_{c\sigma}(t' - t)\right] - 1\right)\mathbf{v_y}\right\} - \mathrm{v}_0(t' - t)\hat{\mathbf{v}}_{\mathbf{y}} \tag{4.1-4}$$

4.2 Plasma Dispersion Function:

An integral function that often crops up in problems involving small-amplitude waves propagating through Maxwellian Warm Plasmas is known as the **Plasma Dispersion Function** or the **Fried-Conte function** (Fried & Conte, 1961) who studied the function with exhaustive detailing.

The Plasma Dispersion Function is,

$$Z(\alpha) = \frac{1}{\sqrt{\pi}}\left[\mathrm{P.V}\int_{-\infty}^{\infty} \mathrm{d}\xi \, \frac{\exp(-\xi^2)}{(\xi - \alpha)}\right]$$

which is defined as it is written for $\mathrm{Im}(\alpha) > 0$, and is analytically continued for $\mathrm{Im}(\alpha) \leq 0$.

- $Z(\alpha)$ is the Hilbert transform of the Gaussian function (Fried & Conte, 1961).
- α is usually of the form $\omega\, /\, k v_{T_\sigma}$ and compared the instability frequence to that due to thermal motions.

Properties -

Owing to its importance, here we shall briefly go over various properties of the function,

Note that,

$$\frac{dZ(\alpha)}{d\alpha} = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}\xi \, \frac{e^{-\xi^2}}{(\xi - \alpha)^2}$$

which on integration by parts yields,

$$\mathrm{Z}'(\alpha) = -\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{2\zeta}{\zeta - \alpha}\mathrm{e}^{-\zeta^2} d\zeta = -2(1 + \alpha Z) \qquad (4.2\text{-}1)$$

For $\alpha \to 0$ from the upper half of the complex plane, we get

$$Z(0) = \frac{1}{\sqrt{\pi}}\,\mathrm{PV}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\zeta^2}}{\zeta} d\zeta + \mathrm{i}\pi^{1/2} = \mathrm{i}\pi^{1/2} \qquad (4.2\text{-}2)$$

Integrating the linear differential (4.2-1), which possesses an integrating factor $\exp \alpha^2$, and using the boundary condition (4.2-2), we obtain an alternative expression for the plasma dispersion function:

$$Z(\alpha) = \mathrm{e}^{-\alpha^2}\left(\mathrm{i}\pi^{1/2} - 2\int_0^{\alpha} \mathrm{e}^{x^2} dx\right)$$

Making the substitution $t = ix$ in the integral, and noting that

$$\int_{-\infty}^{0} \mathrm{e}^{-t^2} dt = \frac{\pi^{1/2}}{2}$$

We arrive at the expression that relates the plasma dispersion function to an error function of imaginary argument (Abramowitz & Stegun, 1972),

$$Z(\alpha) = 2\mathrm{i}\mathrm{e}^{-\alpha^2}\int_{-\infty}^{\mathrm{i}\alpha} \mathrm{e}^{-t^2} dt = \mathrm{i}\pi^{1/2}\mathrm{e}^{-\alpha^2}[1 + \mathrm{erf}\,(\mathrm{i}\alpha)]$$

- This expression is particularly useful because the error function is well understood and so those properties can be extended to the Plasma Dispersion function as well.

Asymptotic Expansion

Following (Huba, J. D., 2000), we have the following small and large expansion for some $z \in \mathbb{C}$:

For $|z| \gg 1$, (corresponds to the **adiabatic fluid limit** of $\omega / k \gg v_{T_\sigma}$)

$$Z(z) = \mathrm{i}\sqrt{\pi}\,\sigma\, e^{-z^2} - \frac{1}{z}\left[1 + \frac{1}{2z^2} + \frac{3}{4z^4} + \frac{15}{8z^6} + \mathcal{O}(z^{-8})\right] \tag{4.2-3}$$

where,

$$\sigma = \begin{cases} 0, & y > 1/|x| \\ 1, & y < 1/|x| \\ 2, & y < -1/|x| \end{cases}$$

For $|z| \ll 1$, (corresponds to the **isothermal fluid limit** of $\omega/k \ll v_{T_\sigma}$)

$$Z(z) = \mathrm{i}\sqrt{\pi}\, e^{-z^2} - 2z\left(1 - \frac{2z^2}{3} + \frac{4z^2}{15} - \frac{8z^6}{105} + \mathcal{O}(z^8)\right) \tag{4.2-4}$$

4.3 The Sokhotski-Plemelj formula:

The Sokhotski–Plemelj theorem is a complex analysis theorem that aids in the evaluation of certain integrals. Julian Sochocki proved the theorem in 1868, and Josip Plemelj rediscovered it in 1908 as part of his solution to the Riemann–Hilbert problem.

The formula is essentially a relation between the following generalized functions (also called distributions),

$$\operatorname*{Lim}_{\varepsilon \to 0} \frac{1}{x - a \pm \mathrm{i}\,\varepsilon} = \mathrm{PV}\frac{1}{x-a} \mp \mathrm{i}\pi\delta(x-a)$$

where $\epsilon > 0$ is an infinitesimal real quantity. This identity formally makes sense only when first multiplied by a function $f(x)$ that is assumed to have the following properties and then integrated over a range of x containing the origin.

i. $f(x)$ is smooth and non-singular in a neighbourhood of the a,

ii. $f(x) \to 0$ sufficiently fast as $x \to \pm\infty$ in order that integrals evaluated over the entire real line are convergent.

iii. Moreover, all surface terms at $\pm\infty$ that arise when integrating by parts are assumed to vanish.

An important version for integrals over the real line is,

$$\operatorname*{Lim}_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{f(x)dx}{x - a \pm i\varepsilon} = \mathrm{PV}\int_{-\infty}^{\infty} \frac{f(x)dx}{x-a} \mp i\pi f(a) \tag{4.3-1}$$

where the Cauchy principal value integral is defined as:

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{f(x)dx}{x-a} \equiv \lim_{\delta \to 0}\left\{\int_{-\infty}^{-\delta} \frac{f(x)dx}{x-a} + \int_{\delta}^{\infty} \frac{f(x)dx}{x-a}\right\}$$

4.4 Asymptotic consistency in Cold Limit Expansions

CASE I

We have the dispersion relation (3.1-1),

$$1+\sum_{\sigma}\frac{1}{k^2\lambda_{D\sigma}^2}\left[1+\alpha_{0\sigma}Z(\alpha_{0\sigma})\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}I_0(k_\perp^2 r_{L\sigma}^2)\ +\sum_{n=-\infty}^{\infty}\begin{pmatrix}\alpha_{0\sigma}\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}I_n(k_\perp^2 r_{L\sigma}^2)\times\\ \{Z(\alpha_{n\sigma})+Z(\alpha_{-n\sigma})\}\end{pmatrix}\right]=0$$

Using cold limit expansion,

$$\alpha_0 Z(\alpha_n)\approx\alpha_0\left\{-\left[\frac{1}{\alpha_n}+\mathcal{O}(\alpha_n^{-3})\right]+\mathrm{i}\sigma\sqrt{\pi}\mathrm{e}^{-\alpha_n^2}\right\}\approx-\frac{\alpha_0}{\alpha_n}+\mathcal{O}(\alpha_n^{-2})$$

and,

$$\alpha_0[Z(\alpha_n)+Z(\alpha_{-n})]\approx\alpha_0\left[-\frac{1}{\alpha_n}-\frac{1}{\alpha_{-n}}+\mathcal{O}(\alpha_n^{-3})+\mathrm{i}\sigma\sqrt{\pi}\left(\mathrm{e}^{-\alpha_n^2}+\mathrm{e}^{-\alpha_{-n}^2}\right)\right]$$
$$\approx-\frac{2\left(\omega-k_y v_0\right)^2}{\left(\omega-k_y v_0\right)^2-n^2\omega_{c\sigma}^2}+\mathcal{O}(\alpha_n^{-2})$$

to get,

$$1+\sum_{\sigma}\frac{1}{k^2\lambda_{D\sigma}^2}\left[1-\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}I_0(k_\perp^2 r_{L\sigma}^2)-2\left(\omega-k_y v_0\right)^2\sum_{n=-\infty}^{\infty}\frac{\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}I_n(k_\perp^2 r_{L\sigma}^2)}{\left(\omega-k_y v_0\right)^2-n^2\omega_{c\sigma}^2}\right]=0 \qquad (4.4\text{-}1)$$

This has been derived for e.g. in (Janhunen et al., 2018).

CASE II

Starting with the dispersion (3.1-2),

$$1+\sum_{\sigma}\frac{\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}}{k^2\lambda_{D\sigma}^2}\left[I_0(k_\perp^2 r_{L\sigma}^2)\left(1+\alpha_{0\sigma}Z(\alpha_{0\sigma})\right)+\sum_{n=1}^{\infty}I_n(k_\perp^2 r_{L\sigma}^2)\left[2+\alpha_{0\sigma}\left\{Z(\alpha_{n\sigma})+Z(\alpha_{-n\sigma})\right\}\right]\right]$$

and using the expansion to leading order,

$$1+\alpha_0 Z(\alpha_0)=\ 1+\alpha_0\left\{-\left[\frac{1}{\alpha_0}+\frac{1}{2\alpha_0^3}+\mathcal{O}\left(\alpha_0^{-5}\right)\right]+\mathrm{i}\sigma\sqrt{\pi}\mathrm{e}^{-\alpha_0^2}\right\}\sim-\frac{1}{2\alpha_0^2}+\mathcal{O}(\alpha_0^{-4})$$

and,

$$2+\alpha_0[Z(\alpha_n)+Z(\alpha_{-n})]\approx 2+\alpha_0\left[-\frac{1}{\alpha_n}-\frac{1}{\alpha_{-n}}+\mathcal{O}(\alpha_n^{-3})+\mathrm{i}\sigma\sqrt{\pi}\left(\mathrm{e}^{-\alpha_n^2}+\mathrm{e}^{-\alpha_{-n}^2}\right)\right]$$
$$\approx-\frac{2n^2\omega_{c\sigma}^2}{\left(\omega-k_y v_0\right)^2-n^2\omega_{c\sigma}^2}+\mathcal{O}(\alpha_n^{-2})$$

We arrive at (3.1-5),

$$1-\sum_{\sigma}\frac{\mathrm{e}^{-k_\perp^2 r_{L\sigma}^2}}{k^2\lambda_{D\sigma}^2}\left[\begin{matrix}I_0(k_\perp^2 r_{L\sigma}^2)\dfrac{1}{2}\left(\dfrac{k_\parallel v_{T\sigma}}{\omega-k_y v_{\sigma 0}}\right)^2\\ +\displaystyle\sum_{n=1}^{\infty}I_n(k_\perp^2 r_{L\sigma}^2)\left(\frac{2n_\sigma^2\omega_{c\sigma}^2}{\left(\omega-k_y v_{\sigma 0}\right)^2-n_\sigma^2\omega_{c\sigma}^2}\right)\end{matrix}\right]=0 \qquad (4.4\text{-}2)$$

Now clearly (4.4-1) and (4.4-2) are not the same, which is not a surprise since (4.4-2) has a higher order approximation and so is a better and consistent asymptotic approximate. The reason we took the trouble of writing the same dispersion in a different form was to collect together terms containing same asymptotic orders, so that the order in which the asymptotic expansions is to be carried out is explicitly clear.

4.5 Bessel's Functions:

The Bessel's Differential equation of order α is,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0, \qquad \alpha\in\mathbb{C}$$

The only Singular point $x=0$ is a regular and the Frobenius recursion method may be used to find the Bessel's functions.

Asymptotic Behaviours –

The equation may be rewritten as,

$$y''+\frac{y'}{x}+\left(1-\frac{\nu^2}{x^2}\right)y=0$$

And by Method of Dominant Balance (Bender & Orszag, 1999), the asymptotic equivalent for $x\to\infty$ is,

$$y''+y=0$$

Hence, we expect the solutions to oscillate for x large.

4.5.1 Summary of Alternate formalisms –

Although we have only used Bessel's and Modified Bessel's of First kind, here is a table containing various formalisms:

Type	First Kind	Second Kind
Bessel Functions	J_α	Y_α
Modified Bessel Functions	I_α	K_α
Hankel Functions	$H_\alpha^{(1)}=J_\alpha+iY_\alpha$	$H_\alpha^{(2)}=J_\alpha-iY_\alpha$
Spherical Bessel Functions	j_n	y_n
Spherical Hankel Functions	$h_n^{(1)}=j_n+iy_n$	$h_n^{(2)}=j_n-iy_n$

4.5.2 Bessel's Identities:

Here is a summary of the various Bessel's Identity used through the text (Arfken et al., 2013; Blinder, 2013; Watson, 1995),

Defining relation for Bessel's function of first kind –

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}z\sin\theta - \mathrm{i}n\theta}\mathrm{d}\theta, \qquad \mathrm{e}^{\mathrm{i}z\sin\,\theta} = \sum_{n=-\infty}^{\infty} J_n(z)\mathrm{e}^{\mathrm{i}n\theta} \tag{4.5-1}$$

Relations involving Modified Bessel's of First kind -

$$\sum_{k=-\infty}^{\infty} I_k(x)t^k = \exp\left\{\frac{1}{2}x\left(t + \frac{1}{t}\right)\right\} \tag{4.5-2}$$

Asymptotic forms of I_α,

$$\lim_{\lambda>>1} I_n(\lambda) = \mathrm{e}^{\lambda}, \qquad \lim_{\lambda<<1} I_n(\lambda) = \frac{1}{n!}\left(\frac{\lambda}{2}\right)^n \tag{4.5-3}$$

Relation between Bessel's and Modified Bessel's of first kind

$$\int_0^{\infty} zJ_n^2(\beta z)\mathrm{e}^{-\alpha^2 z^2}\mathrm{d}z = \frac{1}{2\alpha^2}\mathrm{e}^{-\frac{\beta^2}{2\alpha^2}} I_n\left(\frac{\beta^2}{2\alpha^2}\right) \tag{4.5-4}$$

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