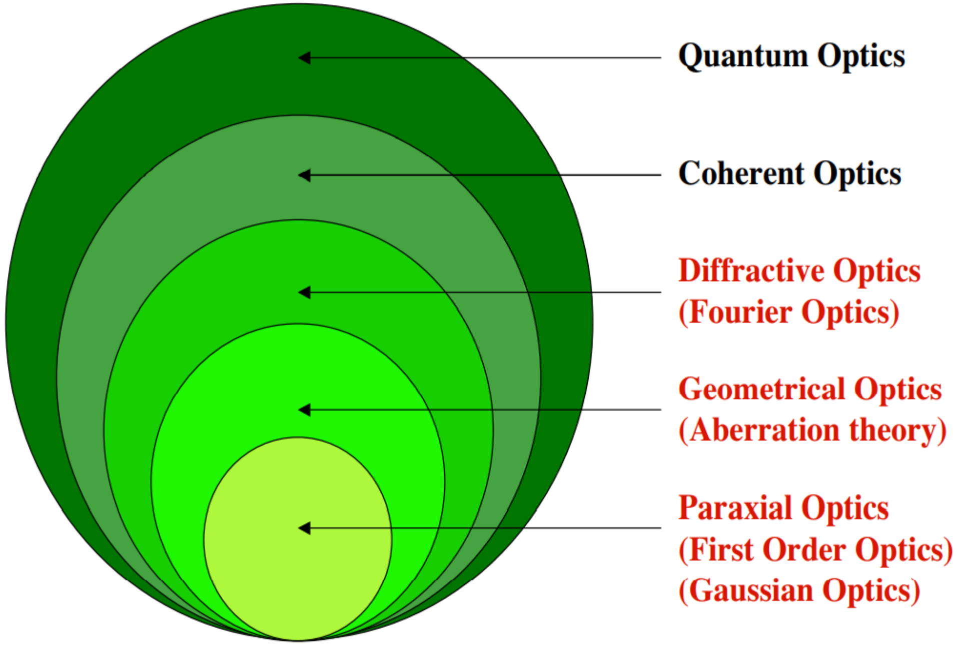
Optics and Introduction to Geometrical Quantization

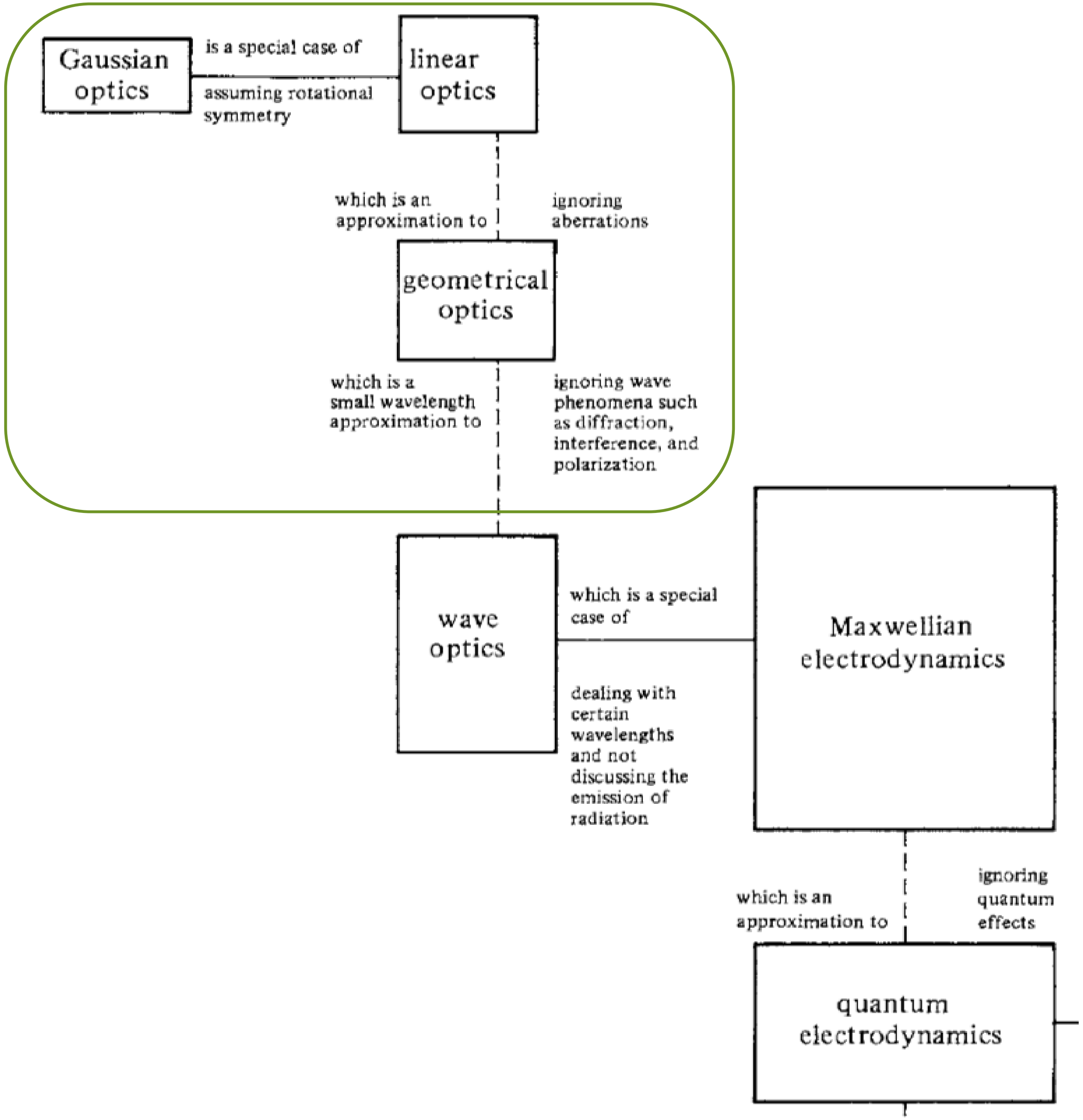
JOVI K, READING PROJECT,

20^{TH} SEP 2021

Bird's Eye view of Optics

AIM – Time evolution / Equation of motion of Light.





Gaussian Optics

- Ray Optics in Paraxial Approximation

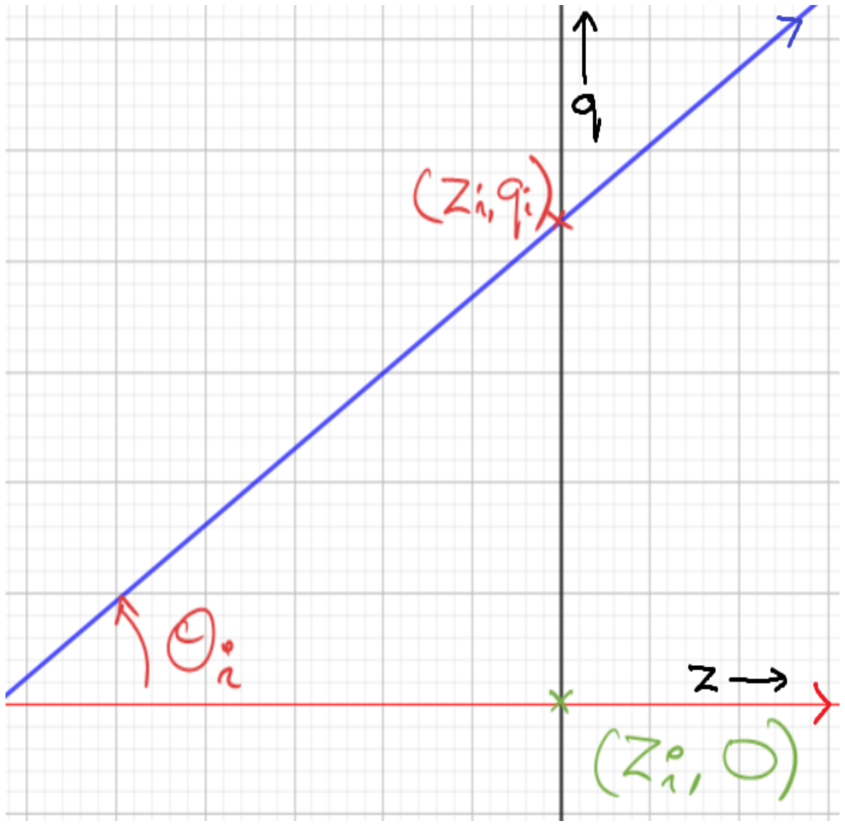
Spherically symmetric rays are specified by $(q_i, p_i = n\theta_i)$ on the z_i plane.

AIM – Given a ray on (q_1, p_1) in z_1 find (q_2, p_2) as a function of q_1, p_1 in z_2.

In linear approximation we have,

$$\begin{pmatrix} q_2 \\ p_2 \end{pmatrix} = M_{21} \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}$$

Note that, $M_{n1} = M_{n\,n-1} \dots M_{21} \Rightarrow \det \mathrm{M} = 1$



Eikonal and Optical Length

Define,

$$\boldsymbol{W = W(q_1, q_2) = (1/2B)\left(Aq_1^2 + Dq_2^2 - 2q_1q_2\right) + K}$$

with $p_1 = -(\partial W/\partial q_1)$ and $p_2 = \partial W/\partial q_2$

By an appropriate choice of the constant K, we can arrange $W(q_1, q_2)$ be the "optical length" of the light ray joining q_1 to q_2.

$$\boldsymbol{L = L_{axis} + 1/2(p_2q_2 - p_1q_1)}$$

Fermat’s Principle

*Let us fix q_1 and q_2 and consider the set of all paths joining q_1 to q_2 that consist of two segments - from $q_1$to q and from q to q_2. Among all such paths, the actual light ray can be characterized as that path for which the optical length L takes on an **extreme value**, that is, for which,*

$$\frac{\partial L}{\partial q} = 0$$

where,

$$L = n_1 d_1 + n_2 d_2 + \frac{1}{2}\{(n_1 / d_1)(q - q_1)^2 + (n_2 / d_2)(q_2 - q)^2 - Pq^2\}$$

We get a minimum if the conjugate plane to z_1 does not lie between z_1 and z_2 and a maximum otherwise.

Gaussian Optics

Any 2 x 2 matrix with determinant 1 can arise as the matrix of some optical system i.e., there is an *isomorphism between* $\boldsymbol{Sl}(\mathbf{2},\mathbb{R})$ *and Gaussian optics*.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}}_{\text{Translation}} \underbrace{\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}}_{\text{Refraction}} \underbrace{\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}}_{\text{Translation}}$$

And $\det M = 1$, where $A + sC = 1$ & $t = -(B + sD)$.

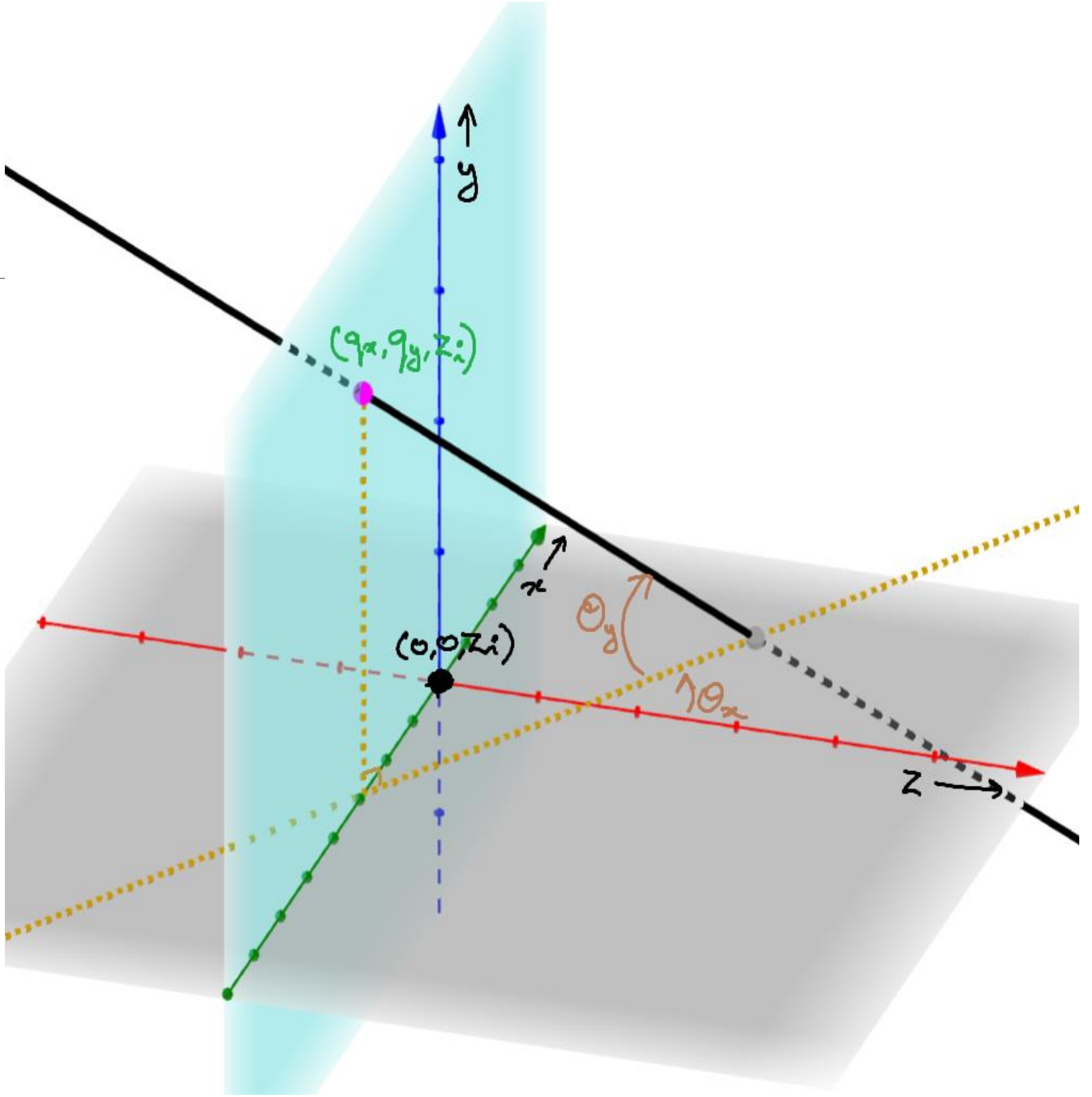
Linear Optics

A ray passing through the xy plane (given z_i) may be specified as,

$$u_1 = \begin{pmatrix} q_{x_1} \\ q_{y_2} \\ p_{x_1} \\ p_{y_2} \end{pmatrix}$$

Generalizing Gaussian Optics,

Does all $Sl(4, \mathbb{R})$ represent a transformation matrices in linear optics ?



Linear Optics

Does all $Sl(4, \mathbb{R})$ represent a transformation matrices in linear optics ?

Nope, BUT,

Linear optics is equivalent to $Sp(4, \mathbb{R})$, the group of linear **symplectic transformations**.

Hence, more generally we can show that,

The manifold M of all light rays carries a natural symplectic structure.

Digression I – Symplectic Transformations

Digression I – Symplectic Transformations

Define a **symplectic differential 2-form** on a real vector space,

$$\Omega : V \times V \to \mathbb{R}$$

i.e.,

- **Antisymmetric:** $\Omega(u, v) = -\Omega(v, u); \forall\, u, v \in V$
- **Bilinear:** linear in each variable when the other variable is held fixed
- **Non degenerate:** $\Omega(u, \_)$ is not identically zero unless u itself is zero.

A **symplectic transformation T** on (V, Ω) is such that $\boldsymbol{\Omega(Tu, Tu') = \Omega(u, u')}$

Or equivalently, $T^*\Omega = \Omega$.

The group of all n-dim Linear symplectic transformations form $\boldsymbol{Sp(n, \mathbb{R})}$

Linear Optics

Does all $Sl(4, \mathbb{R})$ represent a transformation matrices in linear optics ?

Linear optics is equivalent to $Sp(4, \mathbb{R})$, the group of linear symplectic transformations

The Proof has two parts –

i. Physical part – refraction at surface represented by $\begin{pmatrix} I & 0 \\ -P & I \end{pmatrix}$ where $P = P^t$ (with any P) and

– motion in a medium of constant index of refraction by $\begin{pmatrix} I & dI \\ 0 & I \end{pmatrix}$ where d is the optical distance along the axis

ii. Mathematical part – every symplectic matrix can be written as a product of matrices of the above types

The manifold of all light rays carries a natural symplectic structure – Geometric Optics

Redefine $u_1 = \begin{pmatrix} q_x \\ q_y \\ n\theta_x \\ n\theta_y \end{pmatrix}$ as $u_1 = \begin{pmatrix} q_x \\ q_y \\ n\sin\theta_x \\ n\sin\theta_y \end{pmatrix}$ where $n \equiv n(x,y,z)$

OR with appropriate choice of the xy plane we may parametrize the ray as,

$$q_{x_1} = x(z)\,;\; q_{y_2} = y(z)\,; p_x = n\sin\theta_x = \frac{\dot{x}n(x,y,z)}{\sqrt{1+\dot{x}^2+\dot{y}^2}}\,; p_x = \frac{\dot{y}n(x,y,z)}{\sqrt{1+\dot{x}^2+\dot{y}^2}}$$

Then, the optical path is,

$$\boldsymbol{J}'(\gamma) = \int n(x,y,z)(1+\dot{x}^2+\dot{y}^2)^{1/2}dz = \int \boldsymbol{J}(x,y,\dot{x},\dot{y},z)dz \quad \text{where } \dot{x} = dx/dz$$

The manifold of all light rays carries a natural symplectic structure – Geometric Optics

More formally,

For an optical path $\gamma(z) = (x(z), y(z), z)$ parametrized by z-axis, we have the optical path,

$$\boldsymbol{J}'(\gamma) = \int n(x,y,z)(1+\dot{x}^2+\dot{y}^2)^{1/2} dz = \int \boldsymbol{J}(x,y,\dot{x},\dot{y},z)dz \quad \text{where } \dot{x} = dx/dz$$

Define the map ϕ: $\left(q_x, q_y, z, p_x, p_y\right) \in \mathbb{R}^5 \to (x, y, z, \dot{x}, \dot{y}) \in \mathbb{R}^5$,

$$q_{x_1} = x(z)\ ;\ q_{y_2} = y(z)\ ;$$

$$p_x = \frac{d\boldsymbol{J}}{d\dot{x}} = n\sin\theta_x = \frac{\dot{x}n(x,y,z)}{\sqrt{1+\dot{x}^2+\dot{y}^2}}\ ; p_y = \frac{d\boldsymbol{J}}{d\dot{y}} = \frac{\dot{y}n(x,y,z)}{\sqrt{1+\dot{x}^2+\dot{y}^2}}$$

And the Legendre Transform function,

$$\boldsymbol{J}(x,y,z,\dot{x},\dot{y}) = p_x\dot{x} + p_y\dot{y} - H\left(q_x, q_y, z, p_x, p_y\right)$$

The manifold of all light rays carries a natural symplectic structure – Geometric Optics

The Tautological 1-form is,

$$\theta = p_x dq_x + p_y dq_y - H dz$$

And so, the symplectic form,

$$d\theta = \Omega = dp_x \wedge dq_x + dp_y \wedge dq_y - dH \wedge dz$$

This is CLOSED and of RANK 4.

By the standard theorem in the calculus of variations (Loomis and Sternberg, 1968, p. 535),

γ is an extremal of optical length if and only if $\phi \circ \dot{\gamma}$ satisfies

$$\iota_\eta d\theta = 0 \quad \text{where} \quad \eta \in T(\phi \circ \dot{\gamma})$$

The manifold of all light rays carries a natural symplectic structure – Geometric Optics

Leads to,

$$\frac{dq_x}{dz} = \frac{\partial H}{\partial p_x} \Rightarrow \frac{dx(\phi^{-1}\nu)}{dz} = \dot{x}$$

$$\frac{dq_y}{dz} = \frac{\partial H}{\partial p_y} \Rightarrow \frac{dy(\phi^{-1}\nu)}{dz} = \dot{y}$$

$$\frac{dp_x}{dz} = -\frac{\partial H}{\partial q_x}$$

$$\frac{dp_y}{dz} = -\frac{\partial H}{\partial q_y}$$

Since $\iota_\eta d\theta = 0$, flow of the field η is a **Symplectic Diffeomorphism**.

The manifold of all light rays carries a natural symplectic structure – Geometric Optics

The basic assertion of geometrical optics is that the transformation from one z-plane to another is a **symplectic diffeomorphism**.

For two symplectic vector spaces (V_1, ω_1) and (V_2, ω_2) of same dimension, a diffeomorphism $\phi: V_1 \xrightarrow{\sim} V_2$ is a symplectomorphism if $\psi^* \omega_2 = \omega_1$.

Alternatively, we may define the $\Gamma_\phi :=$ graph ϕ, in which case the diffeomorphism ϕ is a symplectomorphism if and only if Γ_ϕ is a **Lagrangian submanifold** of $(V_1 \times \bar{V}_2, \Omega)$, i.e. pullback of the embedding map $\eta: V_1 \hookrightarrow V_1 \times V_2$ is zero, $\eta^* \Omega = 0$.

Wave Optics – Interference

Fresnel's integrality condition:

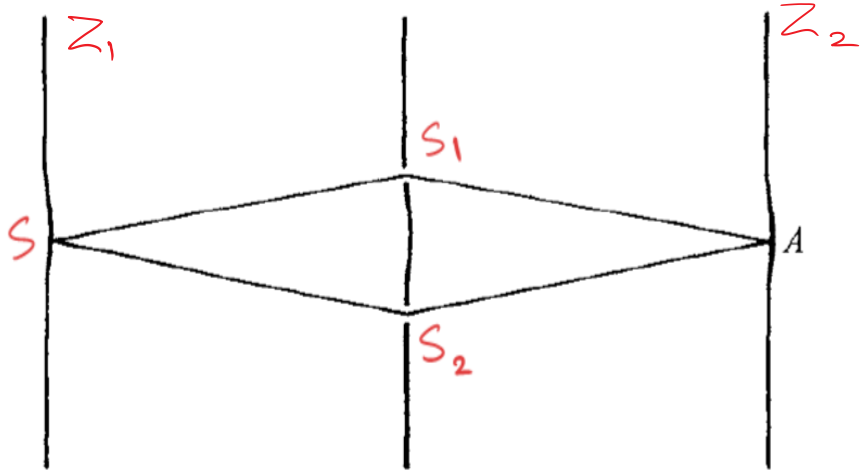
$$\exp 2\pi i L(\gamma_A) \,/\, \lambda = 1.$$

Each point of a light ray can be associated with a complex wavefunction, c.

- where $|c|$ represents the amplitude and decreases due to **attenuation** of light.

  Intensity, $I \propto |c|^2$

- change in phase of c is given by $\exp 2\pi i l \,/\, \lambda$



Young's Double Slit

For different light rays arriving at a point,

$$I \propto |c_1 + c_2 + \cdots|^2$$

Wave Optics

AIM – To find $c_2(q_2), \forall q_2 \in z_2$ given $c_1(q_1), \forall q_1 \in z_1$

PROCEDURE – Find $c_2(q_2)$ for a point q_2 where the wavefunction is the sum of contributions from all q_1. i.e.,

$$c_2(q_2) = \int a(q_1, q_2, \lambda) \exp\left[2\pi i \left(L(q_1, q_2)\right)/\lambda\right] c_1(q_1)\, dq_1$$

Do this for all q_2.

- exp factor represents the change in phase, while
- a is the attenuation factor together with **absolute phase** of the wave.

Wave Optics

1. <u>Magnitude of Attenuation coefficient</u> –

Using the fact that total intensity of the light on the z_2 plane is the same as the total intensity of the light on the z_1 plane. i.e.,

$$\int |c_2(q_2)|^2 dq_2 = \int |c_1(q_1)|^2 dq_1$$

We get,

$$\boldsymbol{|a| = \lambda^{-1} |\det B|^{-1/2}}$$

Wave Optics

1. Magnitude of Attenuation coefficient – $|a| = \lambda^{-1}|det\,B|^{-1/2}$
2. <u>Absolute Phase of Attenuation Coefficient</u> – $a = u|a|$

Principle – At a distance of many wavelengths (i.e., multiples of λ) vertically distant from any stops or diaphragms in our apparatus, the light should behave approximately as if these stops are not present.

$$c(q_1) = \begin{cases} c \text{ for } q_1 \geq 0 \\ 0 \text{ for } q_1 < 0 \end{cases}$$

Wave Optics

1. Magnitude of Attenuation coefficient –

$$|a| = \lambda^{-1}|\det B|^{-1/2}$$

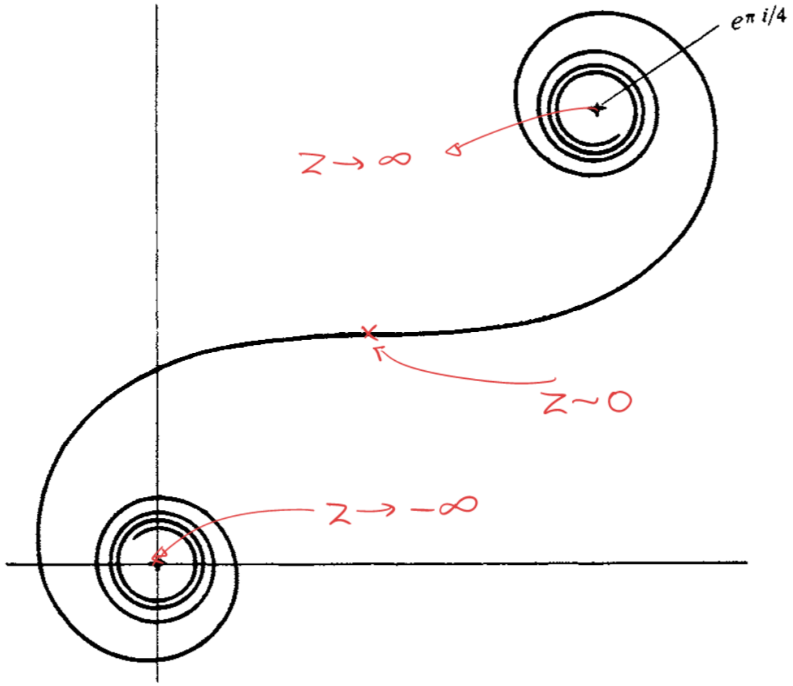
1. <u>Absolute Phase of Attenuation Coefficient</u> –

$$a = u|a|$$

We arrive at the Fresnel Integral,

$$\phi(w) = \int_0^\infty \exp[\pi i (w-r)^2]dr$$

$$\lim_{w\to+\infty} \phi(w) = e^{\pi i/4}\ ; \quad \lim_{w\to-\infty} \phi(w) = 0$$



A double-end Euler / Cornu spiral. The curve continues to converge to the points marked, as w tends to positive or negative infinity.

Wave Optics

1. Magnitude of Attenuation coefficient – $|a| = \lambda^{-1}|det\,B|^{-1/2}$
2. Absolute Phase of Attenuation Coefficient – $a = u|a|$

And hence we have the formula,

$$c(q_2) = \exp(\mp\pi i/4)\,|B\lambda|^{-1/2}\int c(q_1)\exp[2\pi i L(q_1, q_2)/\lambda]dq_1$$

where the phase of a is $u = e^{\mp i\pi/4}$ for $B = \pm|B|$

Fresnel Transformation vs Matric transformation

Gaussian Optics

$$\begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \to M \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} = \begin{pmatrix} Aq_1 + Bp_1 \\ Cq_1 + Dp_1 \end{pmatrix}$$

with the optical matric $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

- works only for single rays

Fresnel Optics

$$c_1 \to c_2 = Fc_1 \equiv (Fc_1)(q_2) =$$

$$\exp(\mp \pi i/4)|B\lambda|^{-1/2} \int c_1(q_1) \times \exp[2\pi i L(q_1, q_2)/\lambda]\, dq_1$$

where $L(q_1, q_2) = \frac{1}{2B}(Aq_1^2 + Dq_2^2 - 2q_1q_2) + d$ for $B \neq 0$

- works for any number of rays hits a point

Fresnel Transformation vs Matric transformation

In the n-dimensional case, $Sp(2n, \mathbb{R})$, where we have the generators,

$$\begin{pmatrix} I & dI \\ 0 & I \end{pmatrix}, d > 0 \qquad (U_d c)(x) = \exp(-\pi i n/4) d^{-n/2} (2\pi)^{-n/2} \int \exp[i(x-y)^2/2d] c(y) dy$$

$$\begin{pmatrix} I & 0 \\ -P & I \end{pmatrix}, P = P^t \qquad (V_{\mathrm{p}} c)(x) = \exp(-iPx \cdot x/2) c(x)$$

Define the map ρ,

$$\rho(U_d) = \begin{pmatrix} I & d \\ 0 & I \end{pmatrix} \text{ and } \rho(V_{\mathrm{P}}) = \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix}$$

For $G = \mathrm{span}\,\{U_d, V_d\}$ we have, a **Homomorphism between $\boldsymbol{G}$ and $\boldsymbol{Sp(2n, \mathbb{R})}$**

Fresnel Transformation vs Matric transformation

For $G = \text{span}\,\{U_d, V_d\}$ we have, a **Homomorphism between $\boldsymbol{G}$ and $\boldsymbol{Sp}(\mathbf{2}\boldsymbol{n}, \mathbb{R})$**

Theorem: If $X \in G$ is such that $\rho(X) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with B nonsingular, then the operator X is given by

$$(Xc)(x) = i^{\#} e^{-\pi i n/4} |\det B|^{-1/2} (2\pi)^{-n/2} \int e^{iW(x,y)} c(y) dy$$

where, $W(x, y) = \frac{1}{2}\left[DB^{-1}x \cdot x + B^{-1}Ay \cdot y - 2B^{t-1}y \cdot x\right]$

$$\# = \begin{cases} \text{even if} \det B > 0 \\ \text{odd if} \det B < 0 \end{cases}$$

Fresnel Transformation vs Matric transformation

For $G = \text{span}\,\{U_d, V_d\}$ we have, a **Homomorphism between $\boldsymbol{G}$ and $\mathrm{Sp}(2\boldsymbol{n}, \mathbb{R})$**

Since **$\boldsymbol{\rho}$ is 2 to 1** everywhere; that is, that $G \equiv \mathrm{Mp}(2n, \mathbb{R})$ is a double covering of the symplectic group, $\mathrm{Sp}(2n, \mathbb{R})$and hence called **Metaplectic group.**

Fresnel optics is equivalent to the study of the metaplectic representation of $Mp\left(4, \mathbb{R}\right)$.

Optics and Mechanics

Analogy between Hamiltonian Mechanics and this Formalism -

1. Co-ordinates –

Just like we require *phase space coordinates* $(\boldsymbol{q}, \boldsymbol{p})$ to completely specify the state of a particle in Mechanics, here too we require (q, p) to specify a light ray (same *symplectic form*).

2. Core Principle –

Both are based on Optimisation Principles – **Variational Principle** in case of Mechanics to find the geodesic while **Fermat's Principle** in Optics for the Optical Trajectory.

Fresnel Optics and Quantum Mechanics - I

For a symplectic manifold $(X = \mathbb{R}^{2n} = \mathbb{R}^n + \mathbb{R}^n, \omega)$ with a symplectomorphism $\phi_t : X \to X$. We have,

$$\phi_t^* \omega = \omega \Rightarrow \frac{d}{dt}\phi_t^*\omega = i(\xi)\, d\omega + d i(\xi)\omega = 0 \Rightarrow \color{red}{di(\xi)\omega = 0 \Rightarrow \boldsymbol{i(\xi)\omega = dH}}$$

For the 2−form, $\omega = -\sum dp_i \wedge dq_i$ and $\xi_H = \sum \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$

We have the **Hamiltonian Equations**,

$$\color{red}{\boldsymbol{\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}; \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}}}$$

Fresnel Optics and Quantum Mechanics - I

Define the Poisson bracket by $\{H_1, H_2\} = -\xi_{H_1} H_2$ or,

$$\Rightarrow \{H_1, H_2\} = \sum \frac{\partial H_1}{\partial q_i}\frac{\partial H_2}{\partial p_i} - \frac{\partial H_2}{\partial q_i}\frac{\partial H_1}{\partial p_i}$$

And,

$$D_{\xi_{H_1}}\left(i(\xi_{H_2})\omega\right) = i\left(D_{\xi_{H_1}}\xi_{H_2}\right)\omega + i(\xi_{H_2})D_{\xi_{H_1}}\omega \quad \textbf{Cartan's Magic Formula}$$

$$\Rightarrow D_{\xi_{H_1}}(dH_2) = -d\{H_1, H_2\} \text{ so } \boldsymbol{\xi_{\{H_1,H_2\}} = -D_{\xi_{H_1}}\xi_{H_2} = \left[\xi_{H_1}, \xi_{H_2}\right]}$$

Fresnel Optics and Quantum Mechanics - I

Homogeneous Quadratic Polynomials -

Quadratic polynomials	Vector fields	Matrix
$\frac{1}{2}p^2$	$p\frac{\partial}{\partial q}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
$\frac{1}{2}q^2$	$-q\frac{\partial}{\partial p}$	$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$
pq	$q\frac{\partial}{\partial q} - p\frac{\partial}{\partial p}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Hamilton's Equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}; \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

ISOMORPHISM between the **algebra of quadratic homogeneous polynomials** (in two variables) and **the symplectic algebra,** $sp(2, \mathbb{R})$.

Fresnel Optics and Quantum Mechanics - II

M(t) is a **one-parameter group** of symplectic matrices i.e.,

$$U(t_1 + t_2) = U(t_1) + U(t_2)$$ and M depends differentiable on t

Take $M'(0) = K \rightarrow M(t) = e^{tK}$

Also, we can have,

$$U'(0)c = \lim_{t \to 0} \frac{U(t)c - c}{t}$$

It is a standard theorem that $U'(0)$ is a skew adjoint operator,

$$\Rightarrow \quad \boldsymbol{\frac{dU}{dt} = -iHU} \quad - \quad \textbf{Schrodinger equation}$$

Fresnel Optics and Quantum Mechanics - II

$\boldsymbol{K}$	$\boldsymbol{M(t)}$	$\boldsymbol{H}$	$\boldsymbol{U(t)}$
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$	$-\frac{1}{2}\frac{d^2}{dx^2}$	$[U(t)f](x) = \exp(\pm\pi i/4)\lvert t\rvert^{-1/2}\int e^{i(x-y)^2/2t} f(y)dy$
$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$	$\frac{1}{2}x^2$	$(U(t)f)(x) = e^{-itx^2/2}f(x)$
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$	$\frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)$	$U(t)f(x) = (-i)^{\#} e^{-\frac{\pi i}{4}}\lvert\sin t\rvert^{-\frac{1}{2}}(2\pi)^{-\frac{1}{2}} \times \int \exp\{\mathrm{i}[\cos t\,(x^2+y^2) - 2xy]/2\sin t\}\, f(y)dy$

The relation between part I and part II are a correspondence between the matric operators $(t \to t\hbar\,/\,m)$, $M(t)$ and Fresnel operators, $U(t)$. For eg: the 1st entry represents a constant change in q with t which corresponds to the Quantum Hamiltonian of free particle.

SYMPLECTIC GEOMETRY

JOVI K, READING PROJECT,

27^{TH} SEP 2021

Skew-symmetric Bilinear Maps

Multilinear Algebra Theorem –

Let Ω be a skew-symmetric bilinear map on finite dimensional V. Then there is a basis $u_1, \dots, u_k, \dots, e_1, \dots, e_n, f_1, \dots, f_n$ of V such that

$$\begin{aligned} \Omega(u_i, v) &= 0, && \text{for all } i \text{ and all } v \in V, \\ \Omega(e_i, e_j) &= 0 = \Omega(f_i, f_j), && \text{for all } i, j, \text{ and} \\ \Omega(e_i, f_j) &= \delta_{ij}, && \text{for all } i, j. \end{aligned}$$

- Not Unique
- $k + 2n = \dim V$; n is invariant of (V, Ω), $2n$ is **rank of Ω**

Symplectic Space

An antisymmetric, nondegenerate bilinear form (non-degenerate 2-form) on V is called a symplectic form.

A symplectic bilinear form is a mapping $\omega : V \times V \to F$ that is

- Bilinear: linear in each argument separately,
- Alternating: $\omega(v, v) = 0$ holds for all $v \in V$, and
- Nondegenerate: $\omega(u, v) = 0$ for all $v \in V$ implies that u is zero. ($\mathbf{\dim V = 2n}$)

A vector space possessing a given symplectic form is called a **symplectic vector space** (V, Ω), or is said to have a **symplectic structure**.

Symplectic Space

Space possessing a given symplectic form is called a **symplectic vector space** (V, Ω), or is said to have a **symplectic structure**.

A symplectic vector space (V, Ω) has a basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ satisfying

$$\Omega(e_i, f_j) = \delta_{ij} \text{and } \Omega(e_i, e_j) = 0 = \Omega(f_i, f_j)$$

Such a basis is called a **symplectic basis** of (V, Ω).

$$\text{Hence,} \quad \Omega(u, v) = [-\ u\ -] \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix} \quad \text{where } u, v \in V \times V$$

Symplectic Transformation

A symplectomorphism ϕ between symplectic vector spaces (V, Ω) and (V', Ω') is a linear isomorphism $\phi : V \xrightarrow{\sim} V$ such that $\phi^* \Omega' = \Omega$.

- By definition of pullback, $(\varphi^* \Omega')(u, v) \Omega' \big(\phi(u), \phi(v)\big)$

If a symplectomorphism exists, (V, Ω) and (V', Ω') are said to be symplectomorphism.

- Equivalence Relation

- May demand condition of Diffeomorphism (on manifolds) in addition

Symplectic Manifold

A manifold M is symplectic if it contains an addition structure of *closed* de-Rham 2-form on M.

A de-Rham 2-form / **exterior 2-form** is a map $\omega_p : T_P M \times T_p M \to \mathbb{R}$ such that for each $p \in M$, ω_p is *skew-symmetric bilinear* on the **tangent space to M at p**, and ω_p varies smoothly in p

Example – For $M = \mathbb{R}^{2n}$ with linear coordinates $x_1, \dots, x_n, \dots, y_1, \dots, y_n$ called **canonical basis** we have the symplectic form,

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

called the **Canonical Symplectic Form**.

Cotangent Bundle as a Symplectic Manifold

Let X be any n-dimensional manifold with coordinate charts $(U, x_1, \dots, x_n)$ for $x \in U$ with $x_i: U \to R$ and $M = T^*X$ its cotangent bundle.

It follows that the differentials $(dx_1)_p, \dots, (dx_n)_p$ form a basis of the cotangent space at p, T_p^*X.

- Transition functions between charts are contravariant transformations

Hence, we have the induced map,

$$T^*U \to \mathbb{R}^{2n}: (x, \zeta) \mapsto (x_1, \dots, x_n, \zeta_1, \dots, \zeta_n)$$

where $x_1, \dots, x_n, \zeta_1, \dots, \zeta_n$ are the cotangent coordinates.

Canonical Forms on Cotangent Bundles

Using the coordinate chart, $T^*U \to \mathbb{R}^{2n}: (x, \zeta) \mapsto (x_1, \dots, x_n, \zeta_1, \dots, \zeta_n)$ define the tautological 1-form α on T^*U by

$$\alpha = \sum_{i=1}^{n} \zeta_i \, dx_i$$

Further, define the canonical 2-form as,

$$\omega = -d\alpha = \sum_{i=1}^{n} dx_i \wedge d\zeta_i$$

- Since α is coordinate independent, ω too is!
- $d\omega = -\,d^2\alpha = 0$, hence a symplectic form.

Canonical Forms on Cotangent Bundles

Property of Tautological 1-form –

THEOREM: For the tautological 1-form on T^*X and a section $s_\mu : X \to T^*X : x \mapsto (x, \mu_x)$. We have,

$$s_\mu^* \alpha = \mu$$

PROOF: Let $V \in T_x\mathrm{X}$ be arbitrary, then (write $\mu_{\mathrm{x}} \in T_x^*\mathrm{X}$)

$$\left(s_\mu^*\alpha\right)_x(V) = \alpha_{(x,\mu_x)}\left(\mathrm{s}_{\mu *}V\right) = \pi^*_{(x,\mu_x)}\mu_{\mathrm{x}}\left(\mathrm{s}_{\mu *}V\right)$$

$$= \mu_{\mathrm{x}}\left(\pi_{(x,\mu_x)*}\mathrm{s}_{\mu *}V\right) = \mu_{\mathrm{x}}\left((\pi \circ s_\mu)_* V\right) = \mu_{\mathrm{x}}(V)$$

$$T^*X \xrightarrow{\pi} X$$

$$TT^*X \xrightarrow{\pi_*} TX$$

$$T^*T^*X \xleftarrow{\pi^*} T^*X$$

Lagrangian Submanifolds

Let (M, ω) be a 2n-dimensional symplectic manifold.

For the inclusion map, $i: Y \hookrightarrow M$, Y is Lagrangian Submanifold if and only if $i^*\omega = 0$ and $\dim Y = 1/2 \dim M$.

Equivalently,

A submanifold Y of M is a Lagrangian Submanifold if, at each $p \in Y$, T_pY is a Lagrangian subspace of T_pM, i.e., $\omega_p\big|_{TpY} = 0$ and $\dim T_pY = 1/2 \dim T_p M$.

Lagrangian subspace on Cotangent Bundles

Example –

Using the coordinate chart, $T^*U \to \mathbb{R}^{2n}: (x,\zeta) \mapsto (x_1,\dots,x_n,\zeta_1,\dots,\zeta_n)$ for $U \subseteq X$ we have the canonical 2-form,

$$\omega = -d\alpha = \sum_{i=1}^{n} dx_i \wedge d\zeta_i$$

The zero section of T^*X is,

$$X_0 \coloneqq \{(x,\zeta) \in T^*X | \zeta = 0 \ \text{ in } \ T^*X\}$$

Hence, $\alpha = \sum_{i=1}^{n} \zeta_i \, dx_i$ vanishes on $X_0 \cap T^*U$ since $i^*\alpha = 0$ for the inclusion $i: X_0 \hookrightarrow T^*X$, and so does ω.

Hence, the **zero section is a Lagrangian submanifold**.

Lagrangian subspace on Cotangent Bundles

There is a one-to-one correspondence between the set of Lagrangian submanifolds of T^*X and the set of closed 1-forms on X.

Lagrangian subspace and Symplectomorphism –

Let (M_1, ω_1) and (M_2, ω_2) be two 2n-dimensional symplectic manifolds with a diffeomorphism $\varphi: M_1 \xrightarrow{\sim} M_2$.

Define the twisted product space $(M_1 \times \overline{M_2}, \widetilde{\omega})$, such that

$$\widetilde{\omega} \coloneqq \pi_1^* \omega_1 - \pi_2^* \omega_2 \ \text{ and } \ M_1 \times \overline{M_2} \in \mathrm{Graph}\, \varphi = \{(p, \varphi(p)) | p \in M_1\}$$

Theorem –

A diffeomorphism φ is a symplectomorphism if and only if Γ_φ is a Lagrangian Submanifold of $(M_1 \times \overline{M_2}, \widetilde{\omega})$.

Darboux Theorem

Let (M, ω) be a symplectic manifold, and let p be any point in M. Then we can find a coordinate system $(U \subset M, x_1, \ldots, x_n, y_1, , \ldots, y_n)$ centered at p such that on U is symplectomorphism to open subset of $\mathbb{R}^{2n} = T^*\mathbb{R}^n$, the 2-forms is,

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

- For any symplectic manifold, the 2-forms look locally the same.

Lagrangian subspace on Cotangent Bundles

Let M be a differentiable manifold, and $\rho: M \times \mathbb{R} \to M$ a map, where we set $\rho_t(p) \coloneqq \rho(p, t)$.

By Picard's theorem , In the neighborhood of any point p and for sufficiently small time t, there is a one-parameter family of local diffeomorphisms ρ_t called **isotopy** satisfying,

$$\frac{\partial \rho_t}{dt} = v_t \circ \rho_t \text{ and } \quad \rho_0 = id_M$$

- One parameter family of diffeomorphism: $\sigma\big(t, \sigma(s + x)\big)1 = \sigma(t + s, x)$

When $v_t = v$ is independent of t, the associated isotopy is called the exponential map or the **flow** and denoted $\exp tv$.

Lagrangian subspace on Cotangent Bundles

Let (M, ω) be a symplectic manifold and let $H: M \rightarrow R$ be a smooth function. Its differential dH is a 1-form. By nondegeneracy, there is a unique vector field X_H on M such that $\iota_{X_H}\omega = dH$.

A vector field X_H, that satisfies $\iota_{X_H}\omega = dH$ is called the **Hamiltonian Vector Field** corresponding to the integral curve H called Hamiltonian Function.

- X is Hamiltonian $\Leftrightarrow$ $\iota_X\omega$ is exact

Hamiltonian Isotopy is a symplectomorphism, since

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{X_H}\omega = \rho_t^*(d\,\underbrace{\iota_{X_H}\omega}_{dH} + \iota_{X_H}\underbrace{d\omega}_{0}) = 0.$$

Classical Mechanics

Consider Euclidean space $\mathbb{R}^{2n}$ with coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ and $\omega_0 = \Sigma dq_j \wedge dp_j$. The curve $\rho_t = (q(t), p(t))$ is an integral curve for X_H exactly if

$$\frac{dq_i}{dt}(t) = \frac{\partial H}{\partial p_i} \; ; \frac{dp_i}{dt}(t) = -\frac{\partial H}{\partial q_i}$$

Indeed for, $X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right)$ we have,

$$\iota_{X_H} \omega = \sum_{j=1}^{n} \iota_{X_H}(dq_j \wedge dp_j) = \sum_{j=1}^{n} \left[(\iota_{X_H} dq_j) \wedge dp_j - dq_j \wedge (\iota_{X_H} dp_j) \right]$$

$$= \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial q_j} dq_j \right) = dH$$

Classical Mechanics

A Hamiltonian system is a triple (M, ω, H), where (M, ω) is a symplectic manifold and $H \in C^{\infty}(M, \mathbb{R})$ is a function, called the Hamiltonian Function.

Theorem – We have $\{f, H\} = 0$ if and only if f is constant along integral curves of X_H, in which case its called an integral of motion (or a first integral or a constant of motion.

Proof – Let ρ_t be the flow of X_H. Then,

$$\frac{d}{dt}(f \circ \rho_t) = \rho_t^* \mathcal{L}_{X_H} f = \rho_t^* \iota_{X_H} df = \rho_t^* \iota_{X_H} \iota_{X_f} \omega$$
$$= \rho_t^* \omega\left(X_f, X_H\right) = \rho_t^* \{f, H\}$$

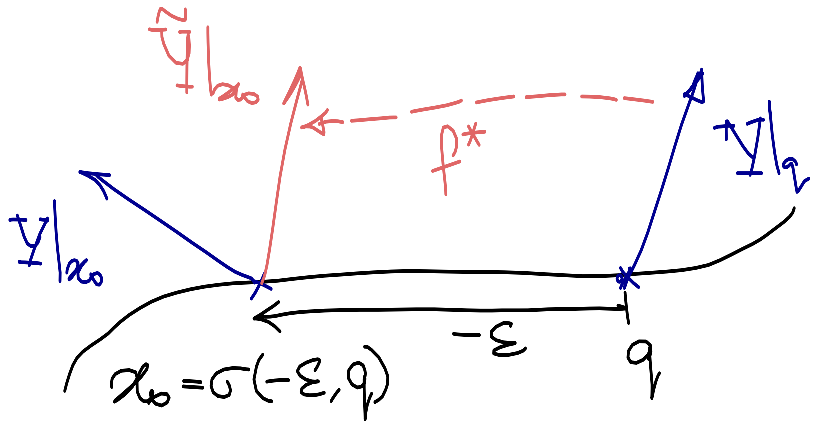
Lie Derivatives

The motivation is the need for a mechanism to compare two objects (vectors/k-forms) in two different spaces.

Lie derivative of two Vectors –

$$\mathcal{L}_X Y(x_0) = \lim_{\varepsilon \to 0} \left[\frac{\tilde{Y}\big|_{x_0} - Y\big|_{x_0}}{\varepsilon} \right]$$

$$= [X, Y]$$



Lie derivative of a scalar –

$$\mathcal{L}_X f(x_0) = X(f(x_0))$$

Lie Bracket

PROPERTIES –

- Antisymmetry: $\mathcal{L}_X Y = [X,Y] = -[Y,X] = -\mathcal{L}_Y X$
- Bilinearity
- Leibnitz rule: $\mathcal{L}_X YZ = Y(\mathcal{L}_X Z) + (\mathcal{L}_X Y)Z$
- Jacobi Identity: $\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0$

$$[\mathcal{L}_X, \mathcal{L}_Y]Z = \big[X,[Y,Z]\big] - \big[Y,[X,Z]\big] = \big[X,[Y,Z]\big] + \big[Y,[Z,X]\big]$$
$$= +\big[Z,[X,Y]\big] = \mathcal{L}_{[X,Y]} Z$$

Cartan's Magic Formula

A relation relating Lie derivative, exterior derivative, and interior product.

Let $\omega \in \Omega^1(m) \coloneqq \omega_\mu \, \mathrm{d}x^\mu$ and $\mathrm{X} = X^\mu \, {}^{\partial}\!/_{\partial x^\mu}$, then ,

$$\mathrm{d}(\iota_X \omega) = \partial_\nu \left(w_\mu X^\mu \right) \mathrm{d}X^\nu$$

And,

$$\iota_X d\omega = \iota_X \, {}^{1}\!/_{2} \left[\partial_\nu \omega_\mu \, dx^\nu \wedge dx^\mu + \partial_\mu \omega_\nu \, dx^\mu \wedge dx^\nu \right]$$
$$= {}^{1}\!/_{2} \left[X^\mu \partial_\nu \omega_\mu - \mathrm{X}^\mu \partial_\mu \omega^\nu \right]$$

Hence,

$$\mathrm{d}(\iota_X \omega) + \iota_X d\omega = \mathcal{L}_X \omega \quad \text{or} \quad \boldsymbol{d\iota_X + \iota_X d = \mathcal{L}_X}$$

Poisson Bracket

The Poisson bracket of two functions $f, g \in C^\infty(M; \mathbb{R})$ is

$$\{f, g\} \coloneqq \omega(X_f, X_g) \xrightarrow{coordinates} \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

And, $X_{\{f,g\}} = -[X_f, X_g]$

A **Poisson algebra** is an associative algebra together with a Lie bracket that also satisfies Leibniz's law.

Complex Vector Space

A complex structure on a vector space, V is a linear map $J: V \to V$ such that $J^2 = -I$ and (V, J) is a **complex vector space**.

For a symplectic vector space (V, Ω), a complex structure J on V is **compactible** (Ω-compactible) if,

$$G_J(u, v) \coloneqq \Omega(u, Jv) \ \ \forall u, v \in V \text{ is a positive inner product on } V$$

Further, if J is a symplectic transformation i.e., $\Omega(Ju, Jv) = \Omega(u, v)$ is called a **Kählerian vector space**.

THEOREM - Let (V, Ω) be a symplectic vector space. Then there is a compatible complex structure J on V.

Almost Complex Structure

An almost complex structure on a manifold M is a smooth field of complex structures on TM,

$$X \in M \mapsto J_x : T_X M \to T_X M \quad \text{linear} \;\&\; \quad J^2 = -I$$

(M, J) is an **almost complex manifold**.

Let (M, ω) be a symplectic manifold. An almost complex structure J on M is called compatible (with ω or ω-compatible) if the assignment,

$$x \mapsto g_x : T_x M \times T_x M \to \mathbb{R}\,; \quad g_x(u, v) := \omega_x(u, J_x v)$$

is a Riemannian metric on M.

Almost Complex Structure

Let (M,ω) be a symplectic manifold, and g a Riemannian metric on M. Then there exists a canonical almost complex structure J on M which is compatible.

Data	**Condition/Technique**	**Consequence**	**Question**
ω, J	$\omega(Ju, Jv) = \omega(u, v)$ $\omega(u, Ju) > 0, u \neq 0$	$g(u, v) := \omega(u, Jv)$ is positive inner product	(g flat?)
g, J	$g(Ju, Jv) = g(u, v)$ (i.e., J is orthogonal)	$\omega(u, v) := g(Ju, v)$ is nondeg., skew-symm.	ω closed?
ω, g	polar decomposition $\rightsquigarrow$	J almost complex str.	J integrable?